Sequentially Cohen-Macaulay Rees modules

Naoki Taniguchi

Meiji University

Joint work with T. N. An, N. T. Dung and T. T. Phuong

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1 Introduction

[CGT]


In [CGT],

- Characterized the sequentially Cohen-Macaulayness of $\mathcal{R}(I)$ where $I$ is an $m$-primary ideal which contains a good parameter ideal as a reduction. ([Theorem 5.3]).

Question 1.1

When is the Rees module $\mathcal{R}(M)$ sequentially Cohen-Macaulay?
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Let $R$ be a Noetherian ring and $M \neq (0)$ a finitely generated $R$-module with $d = \dim_R M < \infty$. We put

$$\Assh_R M = \{p \in \Supp_R M \mid \dim R/p = d\}.$$

Then $\forall n \in \mathbb{Z}, \exists M_n$ the largest $R$-submodule of $M$ with $\dim_R M_n \leq n$. Let

$$S(M) = \{\dim_R N \mid N \text{ is an } R\text{-submodule of } M, N \neq (0)\}$$

$$= \{\dim R/p \mid p \in \Ass_R M\}$$

$$= \{d_1 < d_2 < \cdots < d_\ell = d\}$$

where $\ell = \#S(M)$. 
Let $D_i = M_{d_i}$ for $1 \leq \forall i \leq \ell$. We then have a filtration

$$D_0 := (0) \subsetneq D_1 \subsetneq D_2 \subsetneq \ldots \subsetneq D_\ell = M$$

which we call the dimension filtration of $M$. Put $C_i = D_i/D_{i-1}$ for $1 \leq \forall i \leq \ell$. Notice that $\dim_R D_i = \dim_R C_i = d_i$ for $1 \leq \forall i \leq \ell$.

**Definition 2.1 ([Sch, St])**

1. $M$ is a sequentially Cohen-Macaulay $R$-module
   \[ \iff \] $C_i$ is a C-M $R$-module for $1 \leq \forall i \leq \ell$.
2. $R$ is a sequentially Cohen-Macaulay ring
   \[ \iff \] $\dim R < \infty$ and $R$ is a sequentially C-M module over itself.
Example 2.2

Let \((R, m)\) be a Noetherian local ring, \(M \neq (0)\) a finitely generated \(R\)-module with \(d = \dim_R M\). Then

1. \(d = 1 \Rightarrow M\) is sequentially C-M.
2. \(M\) is C-M \(\Rightarrow M\) is sequentially C-M. The converse holds if \(M\) is unmixed.
3. \(R \ltimes M\) is a sequentially C-M ring \(\iff R\) is a sequentially C-M ring and \(M\) is a sequentially C-M \(R\)-module.
Example 2.3 ([Sch])

Let \( R = k[\Delta] \) be the Stanley-Reisner ring of \( \Delta \) over a field \( k \). If \( \Delta \) is shellable, then \( R \) is sequentially C-M.

Example 2.4

Let \( R \) be a Noetherian local ring, \( G \) a finite subgroup of \( \text{Aut} R \). Suppose that the order of \( G \) is invertible in \( R \). If \( R \) is sequentially C-M, then \( R^G \) is sequentially C-M.
Let

\[(0) = \bigcap_{p \in \text{Ass}_R M} M(p)\]

be a primary decomposition of \((0)\) in \(M\), where \(\text{Ass}_R M/M(p) = \{p\}\) for \(\forall p \in \text{Ass}_R M\).

**Fact 2.5 ([Sch])**

The following assertions hold true.

1. \(D_i = \bigcap_{\dim R/p \geq d_{i+1}} M(p)\) for \(0 \leq \forall i < \ell\).
2. \(\text{Ass}_R C_i = \{p \in \text{Ass}_R M \mid \dim R/p = d_i\}\) and \(\text{Ass}_R D_i = \{p \in \text{Ass}_R M \mid \dim R/p \leq d_i\}\) for \(1 \leq \forall i \leq \ell\).
Theorem 2.6 ([GHS])

Let \( \mathcal{M} = \{M_i\}_{0 \leq i \leq t} \) (\( t > 0 \)) be a family of \( R \)-submodules of \( M \) s.t.

1. \( M_0 = (0) \subsetneq M_1 \subsetneq M_2 \subsetneq \ldots \subsetneq M_t = M \) and
2. \( \dim_R M_{i-1} < \dim_R M_i \) for \( 1 \leq \forall i \leq t \).

Assume that \( \text{Ass}_R M_i/M_{i-1} = \text{Ass}_{h_R} M_i/M_{i-1} \) for \( 1 \leq \forall i \leq t \). Then \( t = \ell \) and \( M_i = D_i \) for \( 0 \leq \forall i \leq \ell \).
Proposition 2.7 (NZD characterization)

Let \((R, m)\) be a Noetherian local ring, \(M \neq (0)\) a finitely generated \(R\)-module. Let \(x \in m\) be a NZD on \(M\). Then TFAE.

1. \(M\) is a sequentially C-M \(R\)-module.
2. \(M/xM\) is a sequentially C-M \(R/(x)\)-module and \(\{D_i/xD_i\}_{0 \leq i \leq \ell}\) is the dimension filtration of \(M/xM\).

Proof.

Since \(x \in m\) is a NZD on \(C_i\) and on \(D_i\) for \(1 \leq \forall i \leq \ell\), so that we get a filtration

\[
D_0/xD_0 = (0) \subsetneq D_1/xD_1 \subsetneq \cdots \subsetneq D_\ell/xD_\ell = M/xM.
\]
Remark 2.8

The implication \((2) \Rightarrow (1)\) is not true without the condition that \(\{D_i/xD_i\}_{0 \leq i \leq \ell}\) is the dimension filtration of \(M/xM\).

For example, let \(R\) be a 2-dimensional Noetherian local domain of depth 1 (e.g., Nagata’s bad example). Then \(R/(x)\) is sequentially C-M for \(x \neq 0\), but \(R\) is not sequentially C-M.
Localization of sequentially C-M modules

Theorem 2.9

Suppose that $\dim R/\mathfrak{p} = \dim R_P/\mathfrak{p} R_P$ for $\forall \mathfrak{p} \in \text{Ass}_R M$ and $\forall P \in \text{Max} R$ s.t. $\mathfrak{p} \subseteq P$. Then TFAE.

1. $M$ is a sequentially C-M $R$-module.
2. $M_P$ is a sequentially C-M $R_P$-module for $\forall P \in \text{Supp}_R M$.

Corollary 2.10

Let $R$ be a finitely generated algebra over a field, $M \neq (0)$ a finitely generated $R$-module. Then TFAE.

1. $M$ is a sequentially C-M $R$-module.
2. $M_P$ is a sequentially C-M $R_P$-module for $\forall P \in \text{Supp}_R M$. 
Theorem 2.11

Let $R = \sum_{n \in \mathbb{Z}} R_n$ be a Noetherian $\mathbb{Z}$-graded ring s.t. $(R, \mathcal{M})$ is an $H$-local ring, $M \neq (0)$ a finitely generated graded $R$-module. Then TFAE.

(1) $M$ is a sequentially C-M $R$-module.

(2) $M_{\mathcal{M}}$ is a sequentially C-M $R_{\mathcal{M}}$-module.

When this is the case, $M_p$ is a sequentially C-M $R_p$-module for $\forall p \in \text{Supp}_R M$. 

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§3 Filtrations of ideals and modules

Let \( R \) be a commutative ring.

**Definition 3.1**

\[ \mathcal{F} = \{ F_n \}_{n \in \mathbb{Z}} \] is a filtration of ideals of \( R \)

\[ \begin{align*}
1 & \quad F_n \text{ is an ideal of } R, \\
2 & \quad F_n \supseteq F_{n+1} \text{ for } \forall n \in \mathbb{Z}, \\
3 & \quad F_m F_n \subseteq F_{m+n} \text{ for } \forall m, n \in \mathbb{Z} \text{ and} \\
4 & \quad F_0 = R.
\end{align*} \]

Then we put

\[ \mathcal{R} = \mathcal{R}(\mathcal{F}) = \sum_{n \geq 0} F_n t^n \subseteq R[t], \quad \mathcal{R}' = \mathcal{R}'(\mathcal{F}) = \sum_{n \in \mathbb{Z}} F_n t^n \subseteq R[t, t^{-1}]. \]
Let $M$ be an $R$-module.

**Definition 3.2**

$\mathcal{M} = \{M_n\}_{n \in \mathbb{Z}}$ is an $\mathcal{F}$-filtration of $R$-submodules of $M$

\[ M_n \text{ is an } R\text{-submodule of } M, \]
\[ M_n \supseteq M_{n+1} \text{ for } \forall n \in \mathbb{Z}, \]
\[ F_m M_n \subseteq M_{m+n} \text{ for } \forall m, n \in \mathbb{Z} \text{ and } \]
\[ M_0 = M. \]

We set

\[ \mathcal{R}(\mathcal{M}) = \sum_{n \geq 0} t^n \otimes M_n \subseteq R[t] \otimes_R M, \]
\[ \mathcal{R}'(\mathcal{M}) = \sum_{n \in \mathbb{Z}} t^n \otimes M_n \subseteq R[t, t^{-1}] \otimes_R M. \]
Here

$$t^n \otimes M_n = \{ t^n \otimes x \mid x \in M_n \} \subseteq R[t, t^{-1}] \otimes_R M$$

for $\forall n \in \mathbb{Z}$.

If $F_1 \neq R$, then we put

$$G = G(F) = \mathcal{R}'/u\mathcal{R}', \quad G(M) = \mathcal{R}'(M)/u\mathcal{R}'(M)$$

where $u = t^{-1}$.
For the rest of this section, we assume that $F_1 \neq R$.

**Lemma 3.3**

Suppose that $R$ is Noetherian and $M$ is finitely generated. Then TFAE.

1. $R(M)$ is a finitely generated graded $R$-module.
2. $R'(M)$ is a finitely generated graded $R'$-module.
3. $\exists n_1, n_2, \ldots, n_\ell \geq 0$ ($\ell > 0$) s.t. $M_n = \sum_{i=1}^{\ell} F_{n-n_i} M_{n_i}$ for $\forall n \geq \max\{n_1, n_2, \ldots, n_\ell\}$. 
The composite map

$$
\psi : \mathcal{R}(\mathcal{M}) \xrightarrow{i} \mathcal{R}'(\mathcal{M}) \xrightarrow{\varepsilon} \mathcal{G}(\mathcal{M})
$$

is surjective and

$$
\text{Ker } \psi = u\mathcal{R}'(\mathcal{M}) \cap \mathcal{R}(\mathcal{M}) = u[\mathcal{R}(\mathcal{M})]_+
$$

where $[\mathcal{R}(\mathcal{M})]_+ = \sum_{n>0} t^n \otimes M_n$. 
Assumption 3.4

- $\mathcal{R}(\mathcal{F})$ a Noetherian ring
- $\mathcal{R}(\mathcal{M})$ a finitely generated $\mathcal{R}$-module

Then $R$ is Noetherian and $M$ is finitely generated.
Suppose that $M \neq (0)$. Then the following assertions hold true.

1. If $d = \dim_R M < \infty$, then 
   \[
   \dim_R \mathcal{R}(M) = \begin{cases} 
   d + 1 & \text{if } \exists \mathfrak{p} \in \Assh_R M \text{ s.t. } F_1 \not\subseteq \mathfrak{p}, \\
   d & \text{otherwise}.
   \end{cases}
   \]

2. \( \dim_{R'} \mathcal{R}'(M) = \dim_R M + 1 \).

3. If $R$ is a local ring, then $\mathcal{G}(M) \neq (0)$, $\dim_\mathcal{G} \mathcal{G}(M) = \dim_R M$. 

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4 Main results

Notation 4.1

- \((R, \mathfrak{m})\) a Noetherian local ring
- \(M \neq (0)\) a finitely generated \(R\)-module with \(d = \dim_R M\)
- \(\mathcal{F} = \{F_n\}_{n \in \mathbb{Z}}\) a filtration of ideals of \(R\) s.t. \(F_1 \neq R\)
- \(\mathcal{M} = \{M_n\}_{n \in \mathbb{Z}}\) an \(\mathcal{F}\)-filtration of \(R\)-submodules of \(M\)
- \(\mathfrak{a} = \mathcal{R}(\mathcal{F})_+ = \sum_{n > 0} F_n t^n\)
- \(\mathfrak{m}\) a unique graded maximal ideal of \(\mathcal{R}\)
- \(\mathcal{R} = \mathcal{R}(\mathcal{F})\) a Noetherian ring
- \(\mathcal{R}(\mathcal{M})\) a finitely generated \(\mathcal{R}\)-module
Let $1 \leq i \leq \ell$. We set

$$D_i = \{ M_n \cap D_i \}_{n \in \mathbb{Z}}, \quad C_i = \{ [(M_n \cap D_i) + D_{i-1}]/D_{i-1} \}_{n \in \mathbb{Z}}.$$  

Then $D_i$ (resp. $C_i$) is an $\mathcal{F}$-filtration of $R$-submodules of $D_i$ (resp. $C_i$). Look at the exact sequence

$$0 \to [D_{i-1}]_n \to [D_i]_n \to [C_i]_n \to 0$$

of $R$-modules for $\forall n \in \mathbb{Z}$. We then have

$$0 \to \mathcal{R}(D_{i-1}) \to \mathcal{R}(D_i) \to \mathcal{R}(C_i) \to 0$$

$$0 \to \mathcal{R}'(D_{i-1}) \to \mathcal{R}'(D_i) \to \mathcal{R}'(C_i) \to 0 \quad \text{and}$$

$$0 \to \mathcal{G}(D_{i-1}) \to \mathcal{G}(D_i) \to \mathcal{G}(C_i) \to 0.$$
Theorem 4.2

TFAE.

(1) $\mathcal{R}'(\mathcal{M})$ is a sequentially C-M $\mathcal{R}'$-module.

(2) $\mathcal{G}(\mathcal{M})$ is a sequentially C-M $\mathcal{G}$-module and $\{\mathcal{G}(D_i)\}_{0 \leq i \leq \ell}$ is the dimension filtration of $\mathcal{G}(\mathcal{M})$.

When this is the case, $\mathcal{M}$ is a sequentially C-M $\mathcal{R}$-module.

Theorem 4.3

Suppose that $\mathcal{M}$ is a sequentially C-M $\mathcal{R}$-module and $F_1 \not\subseteq p$ for $\forall p \in \text{Ass}_R M$. Then TFAE.

(1) $\mathcal{R}(\mathcal{M})$ is a sequentially C-M $\mathcal{R}$-module.

(2) $\mathcal{G}(\mathcal{M})$ is a sequentially C-M $\mathcal{G}$-module, $\{\mathcal{G}(D_i)\}_{0 \leq i \leq \ell}$ is the dimension filtration of $\mathcal{G}(\mathcal{M})$ and $a(\mathcal{G}(C_i)) < 0$ for $1 \leq \forall i \leq \ell$.

When this is the case, $\mathcal{R}'(\mathcal{M})$ is a sequentially C-M $\mathcal{R}'$-module.
Lemma 4.4 (cf. [CGT])

(1) \( \{ R'(D_i) \}_{0 \leq i \leq \ell} \) is the dimension filtration of \( R'(M) \).

(2) If \( F_1 \nsubseteq p \) for \( \forall p \in \text{Ass}_R M \), then \( \{ R(D_i) \}_{0 \leq i \leq \ell} \) is the dimension filtration of \( R(M) \).
Theorem 4.2

TFAE.

(1) $\mathcal{R}'(\mathcal{M})$ is a sequentially C-M $\mathcal{R}'$-module.
(2) $\mathcal{G}(\mathcal{M})$ is a sequentially C-M $\mathcal{G}$-module and $\{\mathcal{G}(\mathcal{D}_i)\}_{0 \leq i \leq \ell}$ is the dimension filtration of $\mathcal{G}(\mathcal{M})$.

When this is the case, $\mathcal{M}$ is a sequentially C-M $\mathcal{R}$-module.
Proof of Theorem 4.2

Look at the exact sequence

$$0 \to \mathcal{R}'(C_i) \to R[t, t^{-1}] \otimes_R C_i \to X \to 0$$

of graded $\mathcal{R}'$-modules for $1 \leq i \leq \ell$.

Since $\mathcal{R}'(C_i)$ is C-M and $X_u = (0)$, we have $R[t, t^{-1}] \otimes_R C_i$ is C-M.

Therefore $M$ is sequentially C-M, because $C_i$ is C-M.
Towards a proof of Theorem 4.3

Fact 4.5 ([F])

Let $I$ be an ideal of $R$ and $t \in \mathbb{Z}$. Consider the following two conditions.

1. $\exists \ell > 0$ s.t. $I^\ell \cdot H^i_m(M) = (0)$ for $\forall i \neq t$.
2. $M_p$ is a C-M $R_p$-module and $t = \dim_{R_p} M_p + \dim R/p$ for $\forall p \in \text{Supp}_R M$ but $p \not\in I$.

Then the implication $(1) \Rightarrow (2)$ holds true. The converse holds, if $R$ is a homomorphic image of a Gorenstein local ring.
Lemma 4.6 (Key lemma)

Suppose that $H^i_{\mathfrak{m}}(G(\mathcal{M}))$ is finitely graded for $\forall i \neq d$. Then $H^i_{\mathfrak{m}}(\mathcal{R}(\mathcal{M}))$ is finitely graded for $\forall i \neq d + 1$.

Proof of Lemma 4.6 (Sketch)

It is enough to show that

$$\exists \ell > 0 \text{ s.t. } a^\ell \cdot H^i_{\mathfrak{m}}(\mathcal{R}(M)) = (0) \text{ for } i \neq d + 1.$$  

To see this, let $P \in \text{Supp}_{\mathcal{R}} \mathcal{R}(M)$ s.t. $P \nsubseteq \mathfrak{a}$ and $P \subseteq \mathcal{M}$. Then we can check that $\mathcal{R}(M)_P$ is C-M and

$$d + 1 = \dim_{\mathcal{R}_P} \mathcal{R}(M)_P + \dim \mathcal{R}_{\mathfrak{m}}/P\mathcal{R}_{\mathfrak{m}}.$$  

Thanks to Fact 4.5, $H^i_{\mathfrak{m}}(\mathcal{R}(\mathcal{M}))$ is finitely graded.
We set
\[ a(N) = \max\{n \in \mathbb{Z} \mid [H^t_t(N)]_n \neq (0)\} \]
for a finitely generated graded \( R \)-module \( N \) of dimension \( t \).

**Theorem 4.7**

TFAE.

1. \( R(\mathcal{M}) \) is a C-M \( R \)-module and \( \dim_R R(\mathcal{M}) = d + 1 \).
2. \( H^i_{\mathfrak{m}}(G(\mathcal{M})) = [H^i_{\mathfrak{m}}(G(\mathcal{M}))]_{-1} \) for \( \forall i < d \) and \( a(G(\mathcal{M})) < 0 \).

When this is the case, \( [H^i_{\mathfrak{m}}(G(\mathcal{M}))]_{-1} \cong H^i_m(M) \) for \( \forall i < d \).
Corollary 4.8

Suppose that $M$ is a C-M $R$-module. Then TFAE.

1. $\mathcal{R}(M)$ is a C-M $R$-module and $\dim_R \mathcal{R}(M) = d + 1$.
2. $\mathcal{G}(M)$ is a C-M $G$-module and $a(\mathcal{G}(M)) < 0$. 
Theorem 4.3

Suppose that $M$ is a sequentially C-M $R$-module and $F_1 \not\subseteq p$ for $\forall p \in \text{Ass}_R M$. Then TFAE.

(1) $\mathcal{R}(M)$ is a sequentially C-M $R$-module.

(2) $\mathcal{G}(M)$ is a sequentially C-M $G$-module, $\{\mathcal{G}(D_i)\}_{0 \leq i \leq \ell}$ is the dimension filtration of $\mathcal{G}(M)$ and $a(\mathcal{G}(C_i)) < 0$ for $1 \leq \forall i \leq \ell$.

When this is the case, $\mathcal{R}'(M)$ is a sequentially C-M $R'$-module.
Proof of Theorem 4.3

\( \mathcal{R}(\mathcal{M}) \) is a sequentially C-M \( \mathcal{R} \)-module

\[ \iff \mathcal{R}(\mathcal{C}_i) = \mathcal{R}(\mathcal{D}_i) / \mathcal{R}(\mathcal{D}_{i-1}) \text{ is a C-M } \mathcal{R} \text{-module for } 1 \leq \forall i \leq \ell \]

\[ \iff \mathcal{G}(\mathcal{C}_i) \text{ is a C-M } \mathcal{G} \text{-module, } a(\mathcal{G}(\mathcal{C}_i)) < 0 \text{ for } 1 \leq \forall i \leq \ell \]

\[ \iff \mathcal{G}(\mathcal{M}) \text{ is a sequentially C-M } \mathcal{G} \text{-module, } \{\mathcal{G}(\mathcal{D}_i)\}_{0 \leq i \leq \ell} \text{ is the dimension filtration of } \mathcal{G}(\mathcal{M}) \text{ and } a(\mathcal{G}(\mathcal{C}_i)) < 0 \text{ for } 1 \leq \forall i \leq \ell. \]
§5 Sequentially C-M property in $E^\mathbb{N}$

Let $R = \sum_{n \geq 0} R_n$ be a $\mathbb{Z}$-graded ring. We put

$$F_n = \sum_{k \geq n} R_k \quad \text{for } \forall n \in \mathbb{Z}.$$ 

Then $F_n$ is a graded ideal of $R$, $\mathcal{F} = \{F_n\}_{n \in \mathbb{Z}}$ is a filtration of ideals of $R$ and $F_1 := R_+ \neq R$.

Let $E$ be a graded $R$-module with $E_n = (0)$ for $\forall n < 0$. Put

$$E(n) = \sum_{k \geq n} E_k \quad \text{for } \forall n \in \mathbb{Z}.$$ 

Then $E(n)$ is a graded $R$-submodule of $E$, $\mathcal{E} = \{E(n)\}_{n \in \mathbb{Z}}$ is an $\mathcal{F}$-filtration of $R$-submodules of $E$.

Then we have

$$R = G(\mathcal{F}) \quad \text{and} \quad E = G(\mathcal{E}).$$
Assumption 5.1

- $R = \sum_{n \geq 0} R_n$ a Noetherian $\mathbb{Z}$-graded ring
- $E \neq (0)$ a finitely generated graded $R$-module with $d = \dim_R E < \infty$

We set

$$R^\mathbb{h} := \mathcal{R}(\mathcal{F}) \quad \text{and} \quad E^\mathbb{h} := \mathcal{R}(\mathcal{E}).$$
Lemma 5.2

Then the following assertions hold true.

(1) $R^h$ is a Noetherian ring.

(2) $E^h$ is a finitely generated graded $R^h$-module.

(3) $\mathcal{R}'(\mathcal{E})$ is a finitely generated graded $\mathcal{R}'$-module.

(4) Suppose that $\exists \mathfrak{p} \in \text{Assh}_R E \text{ s.t. } F_1 \not\subseteq \mathfrak{p}$. Then $\dim_{R^h} E^h = \dim_R E + 1$.

(5) $\dim_{\mathcal{R}'} \mathcal{R}'(\mathcal{E}) = \dim_R E + 1$. 
Let
\[ D_0 = (0) \subsetneq D_1 \subsetneq \ldots \subsetneq D_\ell = E \]
be the dimension filtration of \( E \). Put \( C_i = D_i / D_{i-1}, d_i = \dim_R D_i \)
for \( 1 \leq \forall i \leq \ell \).

Then \( D_i \) is a \text{graded} \( R \)-submodule of \( E \) for \( 0 \leq \forall i \leq \ell \).

Let \( 1 \leq i \leq \ell \). Then we get the exact sequence
\[
0 \to [D_{i-1}]_n \to [D_i]_n \to [C_i]_n \to 0
\]
of graded \( R \)-modules for \( \forall n \in \mathbb{Z} \).
Therefore

\[ 0 \rightarrow \mathcal{R}(\mathcal{D}_{i-1}) \rightarrow \mathcal{R}(\mathcal{D}_i) \rightarrow \mathcal{R}(\mathcal{C}_i) \rightarrow 0 \]

\[ 0 \rightarrow \mathcal{R}'(\mathcal{D}_{i-1}) \rightarrow \mathcal{R}'(\mathcal{D}_i) \rightarrow \mathcal{R}'(\mathcal{C}_i) \rightarrow 0 \]

and

\[ 0 \rightarrow \mathcal{G}(\mathcal{D}_{i-1}) \rightarrow \mathcal{G}(\mathcal{D}_i) \rightarrow \mathcal{G}(\mathcal{C}_i) \rightarrow 0 \]

of graded modules, where $\mathcal{D}_i = \{[D_i](n)\}_{n \in \mathbb{Z}}$, $\mathcal{C}_i = \{[C_i](n)\}_{n \in \mathbb{Z}}$.

**Lemma 5.3**

1. \( \{\mathcal{R}'(\mathcal{D}_i)\}_{0 \leq i \leq \ell} \) is the dimension filtration of $\mathcal{R}'(\mathcal{E})$.

2. If $F_1 \not\subseteq \mathfrak{p}$ for all $\mathfrak{p} \in \text{Ass}_R \ E$, then $\{\mathcal{R}(\mathcal{D}_i)\}_{0 \leq i \leq \ell}$ is the dimension filtration of $\mathcal{R}(\mathcal{E})$. 
Proposition 5.4

TFAE.

(1) \( R'(\mathcal{E}) \) is a sequentially C-M \( R' \)-module.

(2) \( E \) is a sequentially C-M \( R \)-module.
Lemma 5.5

Suppose \( R_0 \) is a local ring, \( E \) is a C-M \( R \)-module and \( \exists \mathfrak{p} \in \text{Assh}_R E \) s.t. \( F_1 \not\subseteq \mathfrak{p} \). Then TFAE.

1. \( E^\mathfrak{a} \) is a C-M \( R^\mathfrak{a} \)-module.
2. \( \alpha(E) < 0 \).

Proof of Lemma 5.5 (Sketch)

Let \( P = \mathfrak{m}R + R_+ \), where \( \mathfrak{m} \) denotes the maximal ideal of \( R_0 \). Then \( P \supseteq F_1 \) and

\[
E = \mathcal{G}(\mathcal{E}) \cong \mathcal{G}(\mathcal{E}_P), \quad R = \mathcal{G} \cong \mathcal{G}(\mathcal{F}_P)
\]

since \( R_+(E_n/E_{n+1}) = (0) \), \( R_+(F_n/F_{n+1}) = (0) \) for \( \forall n \in \mathbb{Z} \).

The assertion comes from the above isomorphisms.
Theorem 5.6

Suppose that $R_0$ is a local ring, $E$ is a sequentially C-M $R$-module and $F_1 \not\subseteq \mathfrak{p}$ for $\forall \mathfrak{p} \in \text{Ass}_R E$. Then TFAE.

1. $E^{\mathfrak{h}}$ is a sequentially C-M $R^{\mathfrak{h}}$-module.
2. $a(C_i) < 0$ for $1 \leq \forall i \leq \ell$. 
§6 Application –Stanley-Reisner algebras–

Notation 6.1

- $V = \{1, 2, \ldots, n\} \ (n > 0)$ a vertex set
- $\Delta$ a simplicial complex on $V$ s.t. $\Delta \neq \emptyset$
- $\mathcal{F}(\Delta)$ a set of facets of $\Delta$
- $m = \#\mathcal{F}(\Delta) \ (> 0)$ its cardinality
- $S = k[X_1, X_2, \ldots, X_n]$ a polynomial ring over a field $k$
- $I_{\Delta} = (X_{i_1}X_{i_2} \cdots X_{i_r} \mid \{i_1 < i_2 < \cdots < i_r\} \notin \Delta)$
- $R = k[\Delta] = S/I_{\Delta}$ the Stanley-Reisner ring of $\Delta$
Definition 6.2

A simplicial complex $\Delta$ is *shellable*  
\[ \overset{\text{def}}{\iff} \text{either } m = 1 \text{ or } m > 1, \text{ then } \exists F_1, F_2, \ldots, F_m \in \mathcal{F}(\Delta) \text{ s.t.} \]

1. $\mathcal{F}(\Delta) = \{ F_1, F_2, \ldots, F_m \}$
2. $\langle F_1, F_2, \ldots, F_{i-1} \rangle \cap \langle F_i \rangle$ is pure and  
   $\dim \langle F_1, F_2, \ldots, F_{i-1} \rangle \cap \langle F_i \rangle = \dim F_i - 1$ for $2 \leq \forall i \leq m$.

Remark 6.3

If $\Delta$ is shellable, then we can take a shelling order  
$F_1, F_2, \ldots, F_m \in \mathcal{F}(\Delta)$ s.t. $\dim F_1 \geq \dim F_2 \geq \cdots \geq \dim F_m$. 
We now regard \( R = \sum_{n \geq 0} R_n \) as a \( \mathbb{Z} \)-graded ring and put

\[
I_n := \sum_{k \geq n} R_k = m^n \quad \text{for} \quad \forall n \in \mathbb{Z}
\]

where \( m := R_+ = \sum_{n > 0} R_n \). Then \( \mathcal{I} = \{ I_n \}_{n \in \mathbb{Z}} \) is an \( m \)-adic filtration of \( R \) and \( I_1 \neq R \).

**Proposition 6.4**

*If \( \Delta \) is shellable, then \( \mathcal{R}'(m) \) is a sequentially C-M ring.*
Remark 6.5

\[ p \not\supseteq I_1 \text{ for } \forall p \in \text{Ass } R \iff F \neq \emptyset \text{ for } \forall F \in \mathcal{F}(\Delta) \iff \Delta \neq \{\emptyset\}. \]

Theorem 6.6

Suppose that \( \Delta \) is shellable with shelling order \( F_1, F_2, \ldots, F_m \in \mathcal{F}(\Delta) \) s.t. \( \dim F_1 \geq \dim F_2 \geq \cdots \geq \dim F_m \) and \( \Delta \neq \{\emptyset\} \). Then TFAE.

1. \( R(m) \) is a sequentially C-M ring.
2. Either \( m = 1 \) or \( m \geq 2 \), then \( \dim F_i - 1 > \# \mathcal{F}(\Delta_1 \cap \Delta_2) \) for \( 2 \leq \forall i \leq m \), where \( \Delta_1 = \langle F_1, F_2, \ldots, F_{i-1} \rangle \), \( \Delta_2 = \langle F_i \rangle \).
Apply Theorem 6.6, we get the following.

**Corollary 6.7**

Suppose that $\dim F_m > 2$. If $\langle F_1, F_2, \ldots, F_{i-1} \rangle \cap \langle F_i \rangle$ is a simplex for $2 \leq \forall i \leq m$, then $R(m)$ is a sequentially C-M ring.
Example 6.8

Let \( \Delta = \langle F_1, F_2, F_3 \rangle \), where \( F_1 = \{1, 2, 3\} \), \( F_2 = \{2, 3, 4\} \) and \( F_3 = \{4.5\} \). Then \( \Delta \) is shellable with the numbering \( \mathcal{F}(\Delta) = \{F_1, F_2, F_3\} \). Then

\[
\langle F_1 \rangle \cap \langle F_2 \rangle , \quad \langle F_1, F_2 \rangle \cap \langle F_3 \rangle
\]

are simplex, so that \( \mathcal{R}(m) \) is a sequentially C-M ring.
**Example 6.9**

Let $\Delta = \langle F_1, F_2, F_3, F_4 \rangle$, where $F_1 = \{1, 2, 5\}$, $F_2 = \{2, 3\}$, $F_3 = \{3, 4\}$ and $F_4 = \{4, 5\}$. Then $\Delta$ is shellable with the numbering $\mathcal{F}(\Delta) = \{F_1, F_2, F_3, F_4\}$. We put $\Delta_1 = \langle F_1, F_2, F_3 \rangle$, $\Delta_2 = \langle F_4 \rangle$. Then

$$\sharp \mathcal{F}(\Delta_1 \cap \Delta_2) = 2 = \dim F_4 - 1,$$

so that $\mathcal{R}(m)$ is not a sequentially C-M ring by Theorem 6.6.
Thank you very much for your attention!
References


