DERIVATIVES WITH MCQ ALEXANDER PAIRS FOR HANDLEBODY-KNOTS

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ABSTRACT. In this paper, we introduce the \( f \)-derivative for multiple conjugation quandles with an MCQ Alexander pair \( f \), where a multiple conjugation quandle is an algebra whose axioms are motivated from handlebody-knot theory, and an MCQ Alexander pair is a pair of maps related to a linear extension of a multiple conjugation quandle. Using this, we define \( f \)-twisted Alexander matrices for handlebody-knots equipped with their multiple conjugation quandle representations. We show that this matrix produces strictly stronger invariants than the (twisted) Alexander matrix for handlebody-knots.

1. Introduction

The (twisted) Alexander matrix of a knot group is obtained by using the free derivative introduced by Fox [6]. Alexander [1] defined a classical knot invariant, called the Alexander polynomial, which is a generator of the elementary ideal of the Alexander matrix of the knot group. Lin [18] and Wada [22] generalized it to the twisted Alexander polynomial of a knot equipped with a group representation. Topological properties of knots, such as the genus, fiberedness, and so on, appear in the Alexander polynomials, and the properties are extended to the twisted Alexander polynomials [5, 8, 9, 10, etc.]. More generally, by using the free derivative, we can obtain invariants of groups equipped with their group representations.

A quandle [17, 19] is an algebra whose axioms are motivated from knot theory. Oshiro and the first author [16] introduced the \( f \)-derivative for quandles with a pair of maps \( f \) called an Alexander pair. An Alexander pair is a dynamical cocycle [2] corresponding to a linear extension of a quandle. They defined invariants of quandles equipped with their quandle representations by using an \( f \)-derivative. The (twisted) Alexander polynomial for knots can be obtained in their framework.

A multiple conjugation quandle (MCQ) [12] is an algebra whose axioms correspond to the Reidemeister moves for handlebody-knots, where a handlebody-knot is a handlebody embedded in the 3-sphere \( S^3 \), which is a generalization of a knot to higher genera. A handlebody-knot can be also regarded as a quotient structure of a spatial graph. The second author [20] introduced a pair of maps called an MCQ Alexander pair, which is an MCQ version of an Alexander pair. The first author [13] introduced the fundamental MCQ of a handlebody-knot, whose representations correspond to MCQ colorings for the handlebody-knot. However, it is not easy to distinguish two MCQs in general. Then counting the number of MCQ representations of the fundamental MCQ of a handlebody-knot to an MCQ

2020 Mathematics Subject Classification. Primary 57K10, 57K12; Secondary 57K14, 57K31.

Key words and phrases. multiple conjugation quandle; Fox derivative; handlebody-knot; twisted Alexander invariant.
gives an elementary combinatorial invariant, called the MCQ coloring number. In this paper, we introduce the $f$-derivative for MCQs with an MCQ Alexander pair $f$ and define the $f$-twisted Alexander matrix for handlebody-knots equipped with their MCQ representations. Furthermore, we see that this matrix produces strictly stronger invariants than the (twisted) Alexander matrix for handlebody-knots and spatial graphs [15].

This paper is organized as follows. In Section 2, we recall the notions of a quandle and a multiple conjugation quandle (MCQ). In Section 3, we review the notion of an MCQ Alexander pair and give some examples of MCQ Alexander pairs. In Section 4, we recall the notions of an MCQ presentation and the fundamental MCQ of a handlebody-knot. We see that two handlebody-knots equipped with their MCQ representations are equivalent if and only if their fundamental MCQs with the MCQ representations are related by a finite sequence of some transformations. In Section 5, we introduce the $f$-derivative for MCQs with an MCQ Alexander pair $f$. In Section 6, we introduce $f$-twisted Alexander matrices and handlebody-knot invariants derived from the matrices with an MCQ Alexander pair $f$. In Section 7, we show that the (twisted) Alexander matrix for handlebody-knots is recoverable from an $f$-twisted Alexander matrix for some MCQ Alexander pair $f$. In Section 8, calculating our invariants, we distinguish two handlebody-knots whose complements have isomorphic fundamental groups. We emphasize that they can not be distinguished by invariants derived from the (twisted) Alexander matrices for handlebody-knots.

2. Multiple conjugation quandles

A quandle [17, 19] is a non-empty set $Q$ equipped with a binary operation $\triangleleft : Q \times Q \to Q$ satisfying the following axioms:

(Q1) For any $a \in Q$, $a \triangleleft a = a$.

(Q2) For any $a \in Q$, the map $\alpha : Q \to Q$ defined by $\alpha(x) = x \triangleleft a$ is bijective.

(Q3) For any $a, b, c \in Q$, $(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$.

We denote the iterated map $(\alpha^n) : Q \to Q$ by $\alpha^n$ for $n \in \mathbb{Z}$. For quandles $(Q_1, \triangleleft_1)$ and $(Q_2, \triangleleft_2)$, a quandle homomorphism $f : Q_1 \to Q_2$ is defined to be a map from $Q_1$ to $Q_2$ satisfying $f(a \triangleleft_1 b) = f(a) \triangleleft_2 f(b)$ for any $a, b \in Q_1$.

Let $G$ be a group and $n$ an integer. We define a binary operation $\triangleleft$ on $G$ by $a \triangleleft b = b^{-n}ab^n$. Then, $(G, \triangleleft)$ is a quandle. We call it the $n$-fold conjugation quandle of $G$ and denote it by $\text{Conj}_n G$. The 1-fold conjugation quandle of $G$ is called the conjugation quandle of $G$ and denoted by $\text{Conj} G$. We define another binary operation $\triangleleft$ on $G$ by $a \triangleleft b = ba^{-1}b$. Then, $(G, \triangleleft)$ is a quandle. We call it the core quandle of $G$ and denote it by $\text{Core} G$. For a positive integer $n$, we denote by $\mathbb{Z}_n$ the cyclic group $\mathbb{Z}/n\mathbb{Z}$ of order $n$. We define a binary operation $\triangleleft$ on $\mathbb{Z}_n$ by $a \triangleleft b = 2b - a$. Then, $(\mathbb{Z}_n, \triangleleft)$ is a quandle. We call it the dihedral quandle of order $n$ and denote it by $R_n$. Let $Q$ be an $R[t^{\pm 1}]$-module for a commutative ring $R$. We define a binary operation $\triangleleft$ on $Q$ by $a \triangleleft b = ta + (1 - t)b$. Then $Q$ is a quandle, called an Alexander quandle.

We define the type of a quandle $Q$ by

$$\text{type} Q = \min \{ n \in \mathbb{Z}_{> 0} \mid x \triangleleft^n y = x \text{ for any } x, y \in Q \},$$

where we set $\min \emptyset := \infty$ for the empty set $\emptyset$, and $\mathbb{Z}_{> 0}$ denotes the set of positive integers. We note that $(Q, \triangleleft^i)$ is also a quandle for any $i \in \mathbb{Z}$, and any finite quandle...
is of finite type. For a quandle \( Q \), an extension of \( Q \) is a quandle \( \widetilde{Q} \) which has a surjective homomorphism \( f : \widetilde{Q} \to Q \) such that the cardinalities of the inverse images \( f^{-1}(a) \) and \( f^{-1}(b) \) coincide for any \( a, b \in Q \).

Let \( (Q, \triangleleft) \) be a quandle and \( R \) a ring. The pair of maps \( f_1, f_2 : Q \times Q \to R \) is an Alexander pair [2] corresponding to a linear extension of \( f \) surjective homomorphism \( f : Q \to \Lambda \) if \( \triangleleft \) is a disjoint union \( \Lambda = Q \times \Lambda \) of \( Q \) on \( \Lambda \) and \( \triangleleft \) is a binary operation \( \triangleleft \) satisfying:

- For any \( a, b \in Q \), \( f_1(a, a) + f_2(a, a) = 1 \).
- For any \( a, b \in Q \), \( f(a, b) \) is invertible.
- For any \( a, b, c \in Q \),
  
  \[
  f_1(a \triangleleft b, c) f_1(a, b) = f_1(a \triangleleft c, b \triangleleft c) f_1(a, c),
  \]
  
  \[
  f_1(a \triangleleft b, c) f_2(a, b) = f_2(a \triangleleft c, b \triangleleft c) f_1(b, c),
  \]
  
  \[
  f_2(a \triangleleft b, c) = f_1(a \triangleleft c, b \triangleleft c) f_2(a, c) + f_2(a \triangleleft c, b \triangleleft c) f_2(b, c).
  \]

An Alexander pair is a dynamical cocycle [2] corresponding to a linear extension of a quandle. Many examples of Alexander pairs are given in [16].

**Definition 2.1** ([12]). A multiple conjugation quandle (MCQ) \( X \) is a disjoint union of groups \( G_\lambda (\lambda \in \Lambda) \) with a binary operation \( \triangleleft : X \times X \to X \) satisfying the following axioms:

- For any \( a, b \in G_\lambda \), \( a \triangleleft b = b^{-1}ab \).
- For any \( x \in X \) and \( a, b \in G_\lambda \), \( x \triangleleft e_\lambda = x \) and \( x \triangleleft (ab) = (x \triangleleft a) \triangleleft b \), where \( e_\lambda \) is the identity of \( G_\lambda \).
- For any \( x, y, z \in X \), \( (x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z) \).
- For any \( x \in X \) and \( a, b \in G_\lambda \), \( (ab) \triangleleft x = (a \triangleleft x)(b \triangleleft x) \), where \( a \triangleleft x, b \triangleleft x \in G_\mu \) for some \( \mu \in \Lambda \).

In this paper, we often omit parentheses. When doing so, we apply binary operations from left on expressions, except for group operations, which we always apply first. For example, we write \( a \triangleleft_1 b \triangleleft_2 c d \triangleleft_3 e \triangleleft_4 f \triangleleft_5 g \) for \( ((a \triangleleft_1 b) \triangleleft_2 (c d)) \triangleleft_3 ((e \triangleleft_4 f) \triangleleft_5 g) \) simply, where each \( \triangleleft_i \) is a binary operation, and \( c \) and \( d \) are elements of the same group. For an MCQ \( X = \bigsqcup_{\lambda \in \Lambda} G_\lambda \), we denote by \( G_a \) the group \( G_\lambda \) containing \( a \in X \), and we denote by \( e_a \) the identity of \( G_a \).

We remark that an MCQ itself is a quandle. For two MCQs \( X_1 = \bigsqcup_{\lambda \in \Lambda} G_\lambda \) and \( X_2 = \bigsqcup_{\mu \in M} G_\mu \), an MCQ homomorphism \( f : X_1 \to X_2 \) is defined to be a map from \( X_1 \) to \( X_2 \) satisfying \( f(x \triangleleft y) = f(x) \triangleleft f(y) \) for any \( x, y \in X_1 \) and \( f(ab) = f(a)f(b) \) for any \( \lambda \in \Lambda \) and \( a, b \in G_\lambda \). An MCQ homomorphism \( \rho : X_1 \to X_2 \) is also called an MCQ representation of \( X_1 \) to \( X_2 \). We denote by \( \text{Hom}(X_1, X_2) \) the set of MCQ homomorphisms from \( X_1 \) to \( X_2 \). We call a bijective MCQ homomorphism an MCQ isomorphism. When there exists an MCQ isomorphism from \( X_1 \) to \( X_2 \), we call that \( X_1 \) and \( X_2 \) are isomorphic, denoted by \( X_1 \cong X_2 \). Let \( \rho_1 : X_1 \to Y \) and \( \rho_2 : X_2 \to Y \) be MCQ representations. We say \( (X_1, \rho_1) \) and \( (X_2, \rho_2) \) are isomorphic, denoted by \( (X_1, \rho_1) \cong (X_2, \rho_2) \). If there exists an MCQ isomorphism \( f : X_1 \to X_2 \) such that \( \rho_1 = \rho_2 \circ f \). For an MCQ \( X = \bigsqcup_{\lambda \in \Lambda} G_\lambda \), an extension of \( X \) is an MCQ \( \widetilde{X} \) which has a surjective MCQ homomorphism \( f : \widetilde{X} \to X \) such that the cardinalities of the inverse images \( f^{-1}(x) \) and \( f^{-1}(y) \) coincide for any \( x, y \in X \).

We recall the definition of a G-family of quandles. A G-family of quandles is an algebraic system which yields an MCQ.
Definition 2.2 ([14]). Let $G$ be a group with identity element $e$. A $G$-family of quandles is a non-empty set $X$ with a family of binary operations $\triangleleft^g : X \times X \to X \ (g \in G)$ satisfying the following axioms:

- For any $x \in X$ and $g \in G$, $x \triangleleft^g x = x$.
- For any $x, y \in X$ and $g, h \in G$, $x \triangleleft^g y = x$ and $x \triangleleft^{gh} y = (x \triangleleft^g y) \triangleleft^h y$.
- For any $x, y, z \in X$ and $g, h \in G$, $(x \triangleleft^g y) \triangleleft^h z = (x \triangleleft^h z) \triangleleft^{1^{-1}gh} (y \triangleleft^h z)$.

Let $R$ be a ring and $G$ a group with identity element $e$. Let $X$ be a right $R[G]$-module, where $R[G]$ is the group ring of $G$ over $R$. Then $(X, \{\triangleleft^g\}_{g \in G})$ is a $G$-family of quandles, called a $G$-family of Alexander quandles, with $x \triangleleft^g y = xg + y(e - g)$ [14]. Let $(Q, \circ)$ be a quandle and put $k := type Q$. Then $(Q, \{\triangleleft^i\}_{i \in \mathbb{Z}_k})$ is a $\mathbb{Z}_k$-family of quandles, where we put $\mathbb{Z}_\infty := \mathbb{Z}$.

Let $(X, \{\triangleleft^g\}_{g \in G})$ be a $G$-family of quandles. Then $X \times G = \bigsqcup_{x \in X} \{x\} \times G$ is an MCQ with

$$(x, g) \circ (y, h) := (x \triangleleft^g y, h^{-1}gh), \quad (x, g)(x, h) := (x, gh)$$

for any $x, y \in X$ and $g, h \in G$ [12]. We call it the associated MCQ of $(X, \{\triangleleft^g\}_{g \in G})$. The associated MCQ $X \times G$ is an extension of $G$, where we regard $G$ as an MCQ with the conjugation operation.

3. MCQ Alexander pairs

In this section, we recall the notion of MCQ Alexander pairs and give some examples of them. Throughout this paper, we assume that every ring has the multiplicative identity $1 \neq 0$. For a ring $R$, we denote by $R^\times$ the group of units of $R$.

Definition 3.1 ([20]). Let $X = \bigsqcup_{\lambda \in \Lambda} G_\lambda$ be an MCQ and $R$ a ring. The pair $(f_1, f_2)$ of maps $f_1, f_2 : X \times X \to R$ is an MCQ Alexander pair if $f_1$ and $f_2$ satisfy the following conditions:

- For any $a, b \in G_\lambda$,
  $$f_1(a, b) + f_2(a, b) = f_1(a, a^{-1}b).$$  

- For any $a, b \in G_\lambda$ and $x \in X$,
  $$f_1(a, x) = f_1(b, x), \quad f_2(ab, x) = f_2(a, x) + f_1(b \triangleleft a, a^{-1} \triangleleft x) f_2(b, x).$$

- For any $x \in X$ and $a, b \in G_\lambda$,
  $$f_1(x, e_\lambda) = 1, \quad f_1(x, ab) = f_1(x \triangleleft a, b) f_1(x, a), \quad f_2(x, ab) = f_1(x \triangleleft a, b) f_2(x, a).$$

- For any $x, y, z \in X$,
  $$f_1(x \triangleleft y, z) f_1(x, y) = f_1(x \triangleleft y, y \triangleleft z) f_1(x, z), \quad f_1(x \triangleleft y, z) f_2(x, y) = f_2(x \triangleleft y, y \triangleleft z) f_1(y, z), \quad f_2(x \triangleleft y, z) = f_1(x \triangleleft y, y \triangleleft z) f_2(x, z) + f_2(x \triangleleft y, y \triangleleft z) f_2(y, z).$$
As same as an Alexander pair for a quandle, an MCQ Alexander pair corresponds to a linear extension of an MCQ [20]. We note that an MCQ Alexander pair is an Alexander pair. We call \((1, 0)\) the \textit{trivial MCQ Alexander pair}, where 0 and 1 respectively denote the zero map and the constant map that sends all elements of the domain to the multiplicative identity 1 of the ring.

We give some examples of MCQ Alexander pairs. In Section 7, we see that the MCQ Alexander pairs in Examples 3.2 and 3.3 are related to the Alexander matrix and the twisted Alexander matrix for handlebody-knots, respectively.

Let \(R\) be a ring. We denote by \(M(m, n; R)\) the set of \(m \times n\) matrices over \(R\) and by \(GL(n; R)\) the set of \(n \times n\) invertible matrices over \(R\). Let \(G_0\) be the abelian group
\[
\left\langle t_1, \ldots, t_r \mid t_1^{k_1}, \ldots, t_r^{k_r}, [t_i, t_j] \ (1 \leq i < j \leq r) \right\rangle,
\]
where \(k_1, \ldots, k_r \in \mathbb{Z}_{\geq 0}\), and \([t_i, t_j]\) denotes the commutator of \(t_i\) and \(t_j\). We remark that the group ring \(\mathbb{Z}[G_0]\) can be identified with the quotient ring of the Laurent polynomial ring \(\mathbb{Z}[t_1^{\pm 1}, \ldots, t_r^{\pm 1}] / (t_1^{k_1} - 1, \ldots, t_r^{k_r} - 1)\).

**Example 3.2.** We set maps \(f_1, f_2 : G_0 \times G_0 \to R[G_0]\) by
\[
f_1(a, b) = b^{-1}, \ f_2(a, b) = b^{-1}a - b^{-1}.
\]
Then the pair \((f_1, f_2)\) is an MCQ Alexander pair.

**Example 3.3.** We set maps \(f_1, f_2 : GL(k, R[G_0]) \times GL(k, R[G_0]) \to M(k, k; R[G_0])\) by
\[
f_1(a, b) = b^{-1}, \ f_2(a, b) = b^{-1}a - b^{-1}.
\]
Then the pair \((f_1, f_2)\) is an MCQ Alexander pair.

**Example 3.4.** Let \(X\) be an MCQ, \(R\) a ring and \(f : X \to R^\times\) an MCQ homomorphism. We set maps \(f_1, f_2 : X \times X \to R\) by
\[
f_1(x, y) = f(y)^{-1}, \ f_2(x, y) = f(y)^{-1}(f(x) - 1).
\]
Then the pair \((f_1, f_2)\) is an MCQ Alexander pair.

By using the following proposition, we can construct MCQ Alexander pairs from Alexander pairs.

**Proposition 3.5.** Let \((Q, \triangleleft)\) be a quandle and assume \(k := \text{type} Q < \infty\). Let \(R\) be a ring and let \((f_1, f_2)\) be an Alexander pair of maps \(f_1, f_2 : Q \times Q \to R\) satisfying
\[
\prod_{i=1}^{k} f_1(x \triangleleft^{k-i} y, y) = 1 \quad \text{and} \quad \sum_{i=1}^{k} f_1(x, x)^i = 0
\]
for any \(x, y \in Q\). Let \(X := Q \times \mathbb{Z}_k\) be the associated MCQ of a \(\mathbb{Z}_k\)-family of quandles \((Q, \{\triangleleft^i\}_{i \in \mathbb{Z}_k})\). We define maps \(\tilde{f}_1, \tilde{f}_2 : X \times X \to R\) by
\[
\tilde{f}_1((x, a), (y, b)) = \prod_{i=1}^{b} f_1(x \triangleleft^{b-i} y, y), \quad \tilde{f}_2((x, a), (y, b)) = \left(\prod_{i=1}^{b-1} f_1(x \triangleleft^{b-i} y, y)\right) \sum_{j=1}^{a} f_1(x \triangleleft y, x \triangleleft y)^{j-a} f_2(x, y),
\]
where for any $l \in \mathbb{Z}$, we denote by $\mathcal{I}$ the integer satisfying $1 \leq \mathcal{I} \leq k$ and $l \equiv \mathcal{I} \mod k$. Then the pair $(\bar{f}_1, \bar{f}_2)$ is an MCQ Alexander pair.

Proof. We remark that

\[
\prod_{i=1}^{k} f_1(x \triangleleft^{-i} y, y) = 1, \text{ especially } f_1(x, x)^k = 1,
\]

\[
\sum_{i=1}^{k} f_1(x, x)^{l+i} = 0
\]

for any $x, y \in Q$ and $l \in \mathbb{Z}$, and that $l_1 + l_2 \equiv l_1 + l_2 \mod k$ for any $l_1, l_2 \in \mathbb{Z}$. Since $(f_1, f_2)$ is an Alexander pair, we have the following equalities:

\[
f_1(x \triangleleft^c y, x \triangleleft y)^{-a} \left(\prod_{i=1}^{\mathcal{I}-1} f_1(x \triangleleft^{-i} y, y)\right) = \left(\prod_{i=1}^{\mathcal{I}-1} f_1(x \triangleleft^{-i} y, y)\right)^a
\]

(1)

\[
\left(\prod_{i=1}^{\mathcal{I}} f_1((x \triangleleft^c y) \triangleleft^{-i} z, z)\right) \left(\prod_{i=1}^{\mathcal{I}} f_1(x \triangleleft^{-i} y, y)\right)
\]

\[
= \left(\prod_{i=1}^{\mathcal{I}} f_1((x \triangleleft^c y) \triangleleft^{-i} y, y)\right) \left(\prod_{i=1}^{\mathcal{I}} f_1(x \triangleleft^{-i} y, y)\right)
\]

(2)

\[
f_2(x \triangleleft^c z, y \triangleleft^c z) \left(\prod_{i=1}^{\mathcal{I}} f_1(y \triangleleft^{-i} z, z)\right) = \left(\prod_{i=1}^{\mathcal{I}} f_1((x \triangleleft y) \triangleleft^{-i} z, z)\right) f_2(x, y),
\]

(3)

\[
\left(\prod_{j=1}^{\mathcal{I}} f_1((x \triangleleft y) \triangleleft^{-i} z, (x \triangleleft y) \triangleleft^c z)^j - a\right) \left(\prod_{i=1}^{\mathcal{I}} f_1((x \triangleleft y) \triangleleft^{-i} z, z)\right)
\]

\[
= \left(\prod_{i=1}^{\mathcal{I}} f_1((x \triangleleft y) \triangleleft^{-i} z, z)\right) \left(\sum_{j=1}^{\mathcal{I}} f_1((x \triangleleft y, x \triangleleft y)^j - a)\right)
\]

(4)

\[
f_2(x \triangleleft z, y \triangleleft z) \left(\sum_{j=1}^{\mathcal{I}} f_1(y \triangleleft z, y \triangleleft z)^j - b\right)
\]

\[
= \sum_{j=1}^{\mathcal{I}} \left(\prod_{i=0}^{j-b-1} f_1((x \triangleleft^{k^{-i}} y) \triangleleft z, y \triangleleft z)\right) f_2((x \triangleleft^{b-j} y) \triangleleft z, y \triangleleft z)
\]

(5)

for any $a, b, c \in \mathbb{Z}$ and $x, y, z \in Q$. By the above equalities (1)–(5), the axioms (2-ii), (4-i), (4-ii) and (4-iii) in Definition 3.1 hold. It is easy to see that the axioms (1-i), (2-i), (3-i), (3-ii) and (3-iii) in Definition 3.1 hold. Therefore $(\bar{f}_1, \bar{f}_2)$ is an MCQ Alexander pair.

Example 3.6. Let $Q$ be a quandle and assume $k := \text{type } Q < \infty$. Let $R$ be a ring and let $(f_1, f_2)$ be an Alexander pair of maps $f_1, f_2 : Q \times Q \to R[t^\pm 1]/(1 + t + \cdots + t^{k-1})$ defined by $f_1(a, b) = t$ and $f_2(a, b) = 1 - t$. Let $X := Q \times \mathbb{Z}_k$ be the associated MCQ of a $\mathbb{Z}_k$-family of quandles $(Q, \{a^i\}_{i \in \mathbb{Z}_k})$. Then, by Proposition 3.5, we have
the MCQ Alexander pair \((\tilde{f}_1, \tilde{f}_2)\) of maps \(\tilde{f}_1, \tilde{f}_2 : X \times X \to R[t^{\pm 1}]/(1 + t + \cdots + t^{k-1})\) defined by
\[
\tilde{f}_1((x, a), (y, b)) = t^b, \quad \tilde{f}_2((x, a), (y, b)) = t^b (1 - a - 1),
\]
which coincides with the case when we set \(f : X \to R[t^{\pm 1}]/(1 + t + \cdots + t^{k-1})\) by \(f(a, x) = t^b\) in Example 3.4.

**Example 3.7.** Let \(Q := \text{Core} G\) be the core quandle of a group \(G\). Let \(R\) be a ring and let \((f_1, f_2)\) be an Alexander pair of maps \(f_1, f_2 : Q \times Q \to R[G]\) defined by \(f_1(a, b) = -ba^{-1}\) and \(f_2(a, b) = 1 + ba^{-1}\). Since type \(Q = 2\), we have the associated MCQ \(X := Q \times \mathbb{Z}_2\) of a \(\mathbb{Z}_2\)-family of quandles \((Q, \{s^t\}_{t \in \mathbb{Z}_2})\). Then, by Proposition 3.5, we have the MCQ Alexander pair \((\tilde{f}_1, \tilde{f}_2)\) of maps \(\tilde{f}_1, \tilde{f}_2 : X \times X \to R[G]\) defined by
\[
\tilde{f}_1((x, a), (y, b)) = \begin{cases} 1 & \text{if } b = 0, \\ -yx^{-1} & \text{otherwise}, \end{cases}
\]
\[
\tilde{f}_2((x, a), (y, b)) = \begin{cases} 0 & \text{if } a = 0, \\ -1 - xy^{-1} & \text{if } a = 1 \text{ and } b = 0, \\ 1 + yx^{-1} & \text{if } a = 1 \text{ and } b = 1. \end{cases}
\]

**Example 3.8.** Let \(Q\) be the conjugacy class of the symmetric group of order 4 consisting of cyclic permutations of length 4. We define a binary operation \(<\) on \(Q\) by \(a < b = b^{-1}ab\). Then \((Q, <)\) is a quandle. We write the elements of \(Q\) by \(g_1 := (1234), g_2 := (1423), g_3 := (1324), g_4 := (1432), g_5 := (1324), g_6 := (1243)\) and define the map \(f_1 : Q \times Q \to \mathbb{Z}_4[t^{\pm 1}]/(t^3 - 1)\) by
\[
f_1(x, y) = \begin{cases} t & \text{if } (x, y) \in \{(g_1, g_3), (g_1, g_4), (g_2, g_1), (g_2, g_3), (g_2, g_5), (g_3, g_1), (g_3, g_5), (g_3, g_6), (g_4, g_1), (g_4, g_5), (g_5, g_1), (g_5, g_2), (g_5, g_6), (g_6, g_1), (g_6, g_2), (g_6, g_3), (g_6, g_5)\}, \\ t^2 & \text{if } (x, y) \in \{(g_1, g_5), (g_5, g_3)\}, \\ t^3 & \text{if } (x, y) \in \{(g_1, g_6), (g_3, g_2)\}, \\ 1 & \text{otherwise}. \end{cases}
\]

Then the pair \((f_1, 0)\) is an Alexander pair, where 0 denotes the zero map. Since type \(Q = 4\), we have the associated MCQ \(X := Q \times \mathbb{Z}_4\) of the \(\mathbb{Z}_4\)-family of quandles \((Q, \{s^t\}_{t \in \mathbb{Z}_4})\). Then, by Proposition 3.5, we have the MCQ Alexander pair \((\tilde{f}_1, 0)\), where \(\tilde{f}_1\) is the map \(\tilde{f}_1 : X \times X \to \mathbb{Z}_4[t^{\pm 1}]/(t^3 - 1)\) defined by \(\tilde{f}_1((x, a), (y, b)) = \prod_{k=1}^{5} f_1(x < b^{-1} \cdot y, y)\).

**Remark 3.9.** In general, a quandle 2-cocycle \(\theta : Q \times Q \to A\) induces an Alexander pair \((f_\theta, 0)\), where \(Q\) and \(A\) denote a quandle and an abelian group, respectively, and \(f_\theta : Q \times Q \to R[A]\) is the map defined by \(f_\theta(a, b) = \theta(a, b)\) for a ring \(R\). The Alexander pair \((f_1, 0)\) in Example 3.8 is obtained through this process from a quandle 2-cocycle (refer to [4, Appendix]). Taniguchi [21] showed that, for classical links, the 0-th elementary ideal of the \(f\)-twisted Alexander matrix [16] with an Alexander pair \(f = (f_\theta, 0)\) can be realized from a quandle cocycle invariant [3] with a quandle 2-cocycle \(\theta\).
Let $\rho : X \to Y$ be an MCQ representation. If $(f_1, f_2)$ is an MCQ Alexander pair of maps $f_1, f_2 : Y \times Y \to R$, then $(f_1 \circ (\rho \times \rho), f_2 \circ (\rho \times \rho))$ is also an MCQ Alexander pair.

Proof. This proposition can be verified by direct calculation. \hfill $\square$

4. MCQ presentations and the fundamental MCQ of a handlebody-link

In this section, we review the notions of MCQ presentations and the fundamental MCQ of a handlebody-link. For details see [13].

The free MCQ $F_{MCQ}(S_\Lambda)$ over a given set of pairwise disjoint sets $S_\Lambda = \{S_\lambda \mid \lambda \in \Lambda\}$ is a free object in the category of MCQs. It consists of all MCQ words that can be built from elements of $\bigcup_{\lambda \in \Lambda} S_\lambda$ as described below. For $k \geq 0$, we set

$$W_{MCQ}(S_\lambda; 0) := \bigcup_{\lambda \in \Lambda} S_\lambda,$$

$$W_{MCQ}(S_\lambda; k + 1) := \left\{ a_1^\varepsilon_1 \cdots a_n^\varepsilon_n \mid n \geq 1, \varepsilon_1, \ldots, \varepsilon_n \in \{0, 1, -1\}, a_1, \ldots, a_n \in W_{MCQ}(S_\lambda; k) \right\}$$

$$\cup \{ x \triangleleft y \mid x, y \in W_{MCQ}(S_\lambda; k) \},$$

where $a_1^\varepsilon_1 \cdots a_n^\varepsilon_n$ and $x \triangleleft y$ are symbols, and we put parentheses in appropriate places. For example,

$$a \triangleleft a, a \triangleleft b, b \triangleleft a, b \triangleleft b, a^0, a^{-1}, b^0, b^{-1}, aa, aa^{-1}, a^{-1}aa, bb^{-1}, bb^0b^{-1}$$

are elements of $W_{MCQ}(\{\{a\}, \{b\}\}; 1)$, and

$$(a \triangleleft b) \triangleleft b^{-1}, aa \triangleleft (a \triangleleft b), (a \triangleleft b) \triangleleft aa, b^{-1} \triangleleft (a \triangleleft b), (a \triangleleft b)^0, (b^{-1})^{-1}, (a \triangleleft b)(a \triangleleft b)$$

are elements of $W_{MCQ}(\{\{a\}, \{b\}\}; 2)$. We define $W_{MCQ}(S_\Lambda) := \bigcup_{k=0}^{\infty} W_{MCQ}(S_\lambda; k)$ and call its element MCQ word in $S_\Lambda$. Two MCQ words in $S_\Lambda$ are multiplicable if they represent multiplicable elements of $F_{MCQ}(S_\Lambda)$.

Lemma 4.1 ([13]). Let $S_\Lambda = \{S_\lambda \mid \lambda \in \Lambda\}$ be a set of pairwise disjoint sets. For $w_1, w_2 \in W_{MCQ}(S_\Lambda)$, $w_1$ and $w_2$ represent the same element in $F_{MCQ}(S_\Lambda)$ if and only if $w_1$ and $w_2$ are related by a finite sequence of the following local replacements on MCQ words:

$ab^\varepsilon b^\varepsilon c \leftrightarrow ab^0 c \leftrightarrow ac,$

$$a \triangleleft b \leftrightarrow b^{-1}ab,$$

$$x \triangleleft a^0 \leftrightarrow x, x \triangleleft ab \leftrightarrow (x \triangleleft a) \triangleleft b,$$

$$(x \triangleleft y) \triangleleft z \leftrightarrow (x \triangleleft z) \triangleleft (y \triangleleft z),$$

$$ab \triangleleft x \leftrightarrow (a \triangleleft x)(b \triangleleft x),$$

where $\varepsilon \in \{0, 1, -1\}$ and $a, b, c, x, y, z \in W_{MCQ}(S_\Lambda)$ such that $a, b, c$ are multiplicable.

Every MCQ has a presentation $\langle S_\Lambda \mid R \rangle$, where $S_\Lambda = \{S_\lambda \mid \lambda \in \Lambda\}$ is a set of pairwise disjoint sets, and $R \subseteq F_{MCQ}(S_\Lambda) \times F_{MCQ}(S_\Lambda)$. It is also denoted $\langle S_\Lambda \mid \lambda \in \Lambda \mid R \rangle$. We call $S_\Lambda$ the generating set of $\langle S_\Lambda \mid R \rangle$ and an element of $R$ a relator of $\langle S_\Lambda \mid R \rangle$. A relator $(a, b)$ is also written as $a = b$. For $x \in \bigcup_{\lambda \in \Lambda} S_\lambda$, we use the same symbol $x$ for the element of $\langle S_\Lambda \mid R \rangle$ represented by $x$. A presentation $\langle S_\Lambda \mid R \rangle$ is
called a finite presentation if $\bigcup_{\lambda \in \Lambda} S_\lambda$ and $R$ are finite. For a finitely presented MCQ, we often write
\[
\langle x_{1,1}, \ldots, x_{1,m_1}; \ldots; x_{l,1}, \ldots, x_{l,m_l} \mid r_1, \ldots, r_m \rangle
\]
\[
:= \langle \{ x_{1,1}, \ldots, x_{1,m_1} \}, \{ x_{l,1}, \ldots, x_{l,m_l} \} \rangle \{ r_1, \ldots, r_m \} \rangle.
\]
We define the MCQ isomorphisms
\[
f_{T1-1} : \langle S_\lambda \mid R \rangle \to \langle S_\lambda \mid R, (x, x) \rangle \quad (x \in FMCQ(S_\lambda)),
\]
\[
f_{T1-2} : \langle S_\lambda \mid R, (a, b) \rangle \to \langle S_\lambda \mid R, (a, b), (b, a) \rangle,
\]
\[
f_{T1-3} : \langle S_\lambda \mid R, (a, b), (b, a), (c, a) \rangle \to \langle S_\lambda \mid R, (a, b), (a, c), (c, a) \rangle,
\]
\[
f_{T1-4} : \langle S_\lambda \mid R, (a_1, a_2), (b_1, b_2) \rangle \to \langle S_\lambda \mid R, (a_1, a_2), (b_1, b_2), (a_1 \bowtie a_2 \bowtie b_2) \rangle,
\]
\[
f_{T1-5} : \langle S_\lambda \mid R, (a_1, a_2), (b_1, b_2), (a_1 b_1^{-1}, a_2 b_2^{-1}) \rangle \to \langle S_\lambda \mid R, (a_1, a_2), (b_1, b_2), (a_1 b_1^{-1}, a_2 b_2^{-1}) \rangle.
\]
For finitely presented MCQs $\langle S_\lambda \mid R \rangle$ and $\langle S'_\lambda \mid R' \rangle$, and MCQ representations $\rho : \langle S_\lambda \mid R \rangle \to X$ and $\rho' : \langle S'_\lambda \mid R' \rangle \to X$, we write $(\langle S_\lambda \mid R \rangle, \rho) \sim_T (\langle S'_\lambda \mid R' \rangle, \rho')$ if they are transformed into each other by a finite sequence of the transformations (T1-1)–(T1-5), (T2) and (T3-1). Clearly, if $(\langle S_\lambda \mid R \rangle, \rho) \sim_T (\langle S'_\lambda \mid R' \rangle, \rho')$, then $(\langle S_\lambda \mid R \rangle, \rho) \cong (\langle S'_\lambda \mid R' \rangle, \rho')$.

In the following, we recall the fundamental MCQ of a handlebody-link and its Wirtinger presentation. A handlebody-link [11] is a disjoint union of handlebodies embedded in the 3-sphere $S^3$. A handlebody-knot is a one component handlebody-link. In this paper, we assume that every component of a handlebody-link is of genus at least 1. Two handlebody-links are equivalent if there is an orientation-preserving self-homeomorphism of $S^3$ which sends one to the other. A diagram of a handlebody-link is a diagram of a spatial trivalent graph whose regular neighborhood is the handlebody-link, where a spatial trivalent graph is a finite trivalent
graph embedded in $S^3$. In this paper, a trivalent graph may contain circle components. Two handlebody-links are equivalent if and only if their diagrams are related by a finite sequence of R1–R6 moves depicted in Figure 1 [11].

Let $D$ be a diagram of a handlebody-link. A Y-orientation of $D$ is a collection of orientations of all edges of $D$ without sources and sinks with respect to the orientation as shown in Figure 2, where an edge of $D$ is a piece of a curve each of whose endpoints is a vertex. In this paper, a circle component of $D$ is also regarded as an edge of $D$. We may represent an orientation of an edge by a normal orientation, which is obtained by rotating a usual orientation counterclockwise by $\pi/2$ on a diagram. A vertex of a Y-oriented diagram can be allocated a sign; the vertex is said to have a sign $+1$ or $-1$. The standard convention of the signs is shown in Figure 2. It is known that every diagram has a Y-orientation.

Let $H$ be a handlebody-link represented by a Y-oriented diagram $D$. We denote by $C(D)$, $V(D)$ and $A(D)$ the sets of crossings, vertices and arcs of $D$, respectively. For each $c \in C(D)$, we denote by $v_c$ the over-arc of $c$, and we denote by $u_c$ and $w_c$ the under-arcs of $c$ such that the normal orientation of $v_c$ points from $u_c$ to $w_c$ as illustrated in the left of Figure 3. For each $\tau \in V(D)$, if $\tau$ has a sign $+1$ (resp. $-1$), we denote by $w_\tau$ the arc whose initial (resp. terminal) vertex is $\tau$, and we denote by $u_\tau$ and $v_\tau$ the arcs incident to $\tau$ such that the normal orientation of $w_\tau$ points from $u_\tau$ to $v_\tau$ as illustrated in the center and right of Figure 3. We denote by $\mathcal{A}^i(D)$ the quotient set of $\mathcal{A}(D)$ by the equivalence relation generated by $\bigcup_{c \in V(D)}\{u_\tau, v_\tau, w_\tau\}$, that is, two arcs $x, x' \in \mathcal{A}(D)$ are equivalent if there exist arcs $x_1, x_2, \ldots, x_n \in \mathcal{A}(D)$ such that $x = x_1, x' = x_n$, and that $x_i$ and $x_{i+1}$ have a common vertex of $D$ for each $i$. For example, for the Y-oriented diagram $D$ of a handlebody-knot depicted in Figure 4, we have $\mathcal{A}^i(D) =$
\{ \{x_1, x_2, x_3\}, \{x_4, \ldots, x_{10}\}, \{x_{11}\}, \ldots, \{x_{14}\} \}. \) Then we define
\[
\text{MCQ}(D) := \langle A \sqcup (D) \mid r_c, r_\tau \ (c \in C(D), \tau \in V(D)) \rangle,
\]
(6)
where \( r_c \) and \( r_\tau \) denote the relators \((u_c \triangleleft v_c, w_c)\) and \((u_\tau v_\tau, w_\tau)\), respectively. The isomorphism class of \( \text{MCQ}(D) \) does not depend on the choice of a diagram \( D \) of \( H \) and its Y-orientation [13]. We then define \( \text{MCQ}(H) := \text{MCQ}(D) \) and call it the fundamental \( \text{MCQ} \) of \( H \). The presentation (6) is called the Wirtinger presentation of \( \text{MCQ}(H) \) with respect to \( D \).

Let \( D \) be a Y-oriented diagram of a handlebody-link \( H \) and let \( X \) be an \( \text{MCQ} \). An \( X \)-coloring of \( D \) is a map \( C : A(D) \to X \) satisfying the conditions
\[
C(u_c) \triangleleft C(v_c) = C(w_c) \quad \text{and} \quad C(u_\tau v_\tau) = C(w_\tau)
\]
for each crossing \( c \in C(D) \) and vertex \( \tau \in V(D) \). We denote by \( \text{Col}_X(D) \) the set of \( X \)-colorings of \( D \). An \( X \)-coloring of \( D \) can be regarded as an \( \text{MCQ} \) representation of \( \text{MCQ}(D) \) to \( X \). We can then identify \( \text{Col}_X(D) \) with \( \text{Hom}(\text{MCQ}(D), X) \). Hence its cardinality gives an invariant for the handlebody-link, called the \( \text{MCQ} \) coloring number.

Let \( D \) be a Y-oriented diagram of a handlebody-link \( H \) and \( D' \) a Y-oriented diagram of \( H \) obtained by changing the Y-orientation of \( D \) once. We then obtain a unique \( \text{MCQ} \) isomorphism \( f_{(D,D')} : \text{MCQ}(D) \to \text{MCQ}(D') \) sending \( x \) into \( x^{\varepsilon(x)} \) for any \( x \in A(D) \), where \( \varepsilon(x) = 1 \) if the Y-orientations of \( D \) and \( D' \) coincide on \( x \); otherwise \( \varepsilon(x) = -1 \). Moreover, let \( D'' \) a Y-oriented diagram of \( H \) obtained by applying one of Reidemeister moves preserving the Y-orientation to \( D \) once. We then obtain a unique \( \text{MCQ} \) isomorphism \( f_{(D,D'')} : \text{MCQ}(D) \to \text{MCQ}(D'') \) sending \( x \) into \( x \) for
any $x \in A(D \cap D')$, where $A(D \cap D')$ denotes the set of arcs in the outside of the disk where the move is applied. Let $H$ and $H'$ be handlebody-links represented by $Y$-oriented diagrams $D$ and $D'$, respectively. Let $\rho : MCQ(D) \to X$ and $\rho' : MCQ(D') \to X$ be MCQ representations. Then $(H, \rho)$ and $(H', \rho')$ are equivalent, denoted by $(H, \rho) \cong (H', \rho')$, if there exists a sequence $D = D_1 \leftrightarrow \cdots \leftrightarrow D_n = D'$ of Reidemeister moves and $Y$-orientation changes such that $\rho' = \rho \circ f_{(D_1,D_2)} \circ \cdots \circ f_{(D_{n-1},D_n)}$. If $H \cong H'$, then for any MCQ representation $\rho : MCQ(D) \to X$, there exists a unique MCQ representation $\rho' : MCQ(D') \to X$ such that $(H, \rho) \cong (H', \rho')$. Then we have the following lemma.

**Lemma 4.2.** Let $H$ and $H'$ be handlebody-links represented by $Y$-oriented diagrams $D$ and $D'$, respectively. Let $\rho : MCQ(D) \to X$ and $\rho' : MCQ(D') \to X$ be MCQ representations. If $(H, \rho) \cong (H', \rho')$, then it follows $(MCQ(D), \rho) \sim_T (MCQ(D'), \rho')$.

**Proof.** Assume $(H, \rho) \cong (H', \rho')$. By the definition, there exists a sequence $D = D_1 \leftrightarrow \cdots \leftrightarrow D_n = D'$ of Reidemeister moves and $Y$-orientation changes such that $\rho' = \rho \circ f_{(D_1,D_2)} \circ \cdots \circ f_{(D_{n-1},D_n)}$. By [13], each MCQ isomorphism $f_{(D_i,D_{i+1})}$ can be realized as a composition of $f_{T_{1,1}}^{\pm 1}, f_{T_{1,5}}^{\pm 1}, f_{T_2}^{\pm 1}$ and $f_{T_{3,1}}^{\pm 1}$. Then we have $(MCQ(D), \rho) \sim_T (MCQ(D'), \rho')$.

5. Derivatives with MCQ Alexander pairs

Let $S_\Lambda = \{S_\lambda \mid \lambda \in \Lambda\}$ be a finite set of pairwise disjoint finite sets and $x_1, \ldots, x_n$ the elements of $\bigcup_{\lambda \in \Lambda} S_\lambda$. Let $X = \langle S_\Lambda \mid \{r_1, \ldots, r_m\} \rangle$ be a finitely presented MCQ. Let $F_{MCQ}(S_\Lambda)$ be the free MCQ on $S_\Lambda$ and $pr : F_{MCQ}(S_\Lambda) \to X$ be the canonical projection. We often omit “pr” to represent $pr(x)$ as $x$. Let $f = (f_1, f_2)$ be an MCQ Alexander pair of maps $f_1, f_2 : X \times X \to R$. We denote by $G_{\mu}$ each direct summand of $F_{MCQ}(S_\Lambda)$, where $\mu$ is an element of an index set $\bar{\Lambda}$, that is, $F_{MCQ}(S_\Lambda) = \bigcup_{\mu \in \bar{\Lambda}} G_{\mu}$.

**Definition 5.1.** For $j \in \{1, \ldots, n\}$, the $f$-derivative with respect to $x_j$ is a map $\frac{\partial f}{\partial x_j} : F_{MCQ}(S_\Lambda) \to R$ satisfying

$$\frac{\partial f}{\partial x_j}(x \circ y) = f_1(x, y) \frac{\partial f}{\partial x_j}(x) + f_2(x, y) \frac{\partial f}{\partial x_j}(y),$$

$$\frac{\partial f}{\partial x_j}(ab) = \frac{\partial f}{\partial x_j}(a) + f_1(a, a^{-1}) \frac{\partial f}{\partial x_j}(b),$$

$$\frac{\partial f}{\partial x_j}(x_i) = \delta_{ij}$$

for any $x, y \in F_{MCQ}(S_\Lambda)$, $a, b \in G_{\mu}$ and $i \in \{1, \ldots, n\}$, where $\delta_{ij}$ denotes the Kronecker delta.

**Theorem 5.2.** For $j \in \{1, \ldots, n\}$, the $f$-derivative $\frac{\partial f}{\partial x_j} : F_{MCQ}(S_\Lambda) \to R$ is well-defined.
Proof. We temporarily regard the $f$-derivative $\frac{\partial f}{\partial x_j}$ as the map $\frac{\partial f}{\partial x_j} : W_{MCQ}(S_\Lambda) \to R$ defined by using the following equalities inductively:

\[
\frac{\partial f}{\partial x_j}(x \triangleleft y) = f_1(x, y) \frac{\partial f}{\partial x_j}(x) + f_2(x, y) \frac{\partial f}{\partial x_j}(y),
\]
\[
\frac{\partial f}{\partial x_j}(ab) = \frac{\partial f}{\partial x_j}(a) + f_1(a, a^{-1}) \frac{\partial f}{\partial x_j}(b),
\]
\[
\frac{\partial f}{\partial x_j}(a^{-1}) = -f_1(a, a) \frac{\partial f}{\partial x_j}(a),
\]
\[
\frac{\partial f}{\partial x_j}(x_i) = \delta_{ij},
\]
\[
\frac{\partial f}{\partial x_j}(a^0) = 0
\]

for $x, y, a, b \in W_{MCQ}(S_\Lambda)$, where $a$ and $b$ are multiplicable. This map is well-defined since for any $a, b, c \in W_{MCQ}(S_\Lambda)$ which are multiplicable, we have

\[
\frac{\partial f}{\partial x_j}((ab)c) = \frac{\partial f}{\partial x_j}((ab)) \frac{\partial f}{\partial x_j}(c) + f_1(ab, b^{-1}a^{-1}) \frac{\partial f}{\partial x_j}(c)
\]
\[
= \frac{\partial f}{\partial x_j}(a) + f_1(a, a^{-1}) \frac{\partial f}{\partial x_j}(b) + f_1(ab, b^{-1}a^{-1}) \frac{\partial f}{\partial x_j}(c)
\]
\[
= \frac{\partial f}{\partial x_j}(a) + f_1(a, a^{-1}) \frac{\partial f}{\partial x_j}(bc)
\]
\[
= \frac{\partial f}{\partial x_j}(a(bc)).
\]

For any $x_1, x_2, y_1, y_2, a_1, a_2, b_1, b_2 \in W_{MCQ}(S_\Lambda)$ satisfying that $a_i, b_i$ are multiplicable for $i = 1, 2$, if

\[
\frac{\partial f}{\partial x_j}(x_1) = \frac{\partial f}{\partial x_j}(x_2), \quad \frac{\partial f}{\partial x_j}(y_1) = \frac{\partial f}{\partial x_j}(y_2), \quad \frac{\partial f}{\partial x_j}(a_1) = \frac{\partial f}{\partial x_j}(a_2), \quad \frac{\partial f}{\partial x_j}(b_1) = \frac{\partial f}{\partial x_j}(b_2)
\]

and $x_1 = x_2, y_1 = y_2, a_1 = a_2, b_1 = b_2$ in $F_{MCQ}(S_\Lambda)$, then we have

\[
\frac{\partial f}{\partial x_j}(x_1 \triangleleft y_1) = \frac{\partial f}{\partial x_j}(x_2 \triangleleft y_2), \quad \frac{\partial f}{\partial x_j}(a_1b_1) = \frac{\partial f}{\partial x_j}(a_2b_2).
\]
Hence, by Lemma 4.1, the well-definedness of the $f$-derivative $\frac{\partial f}{\partial x_j} : F_{MCQ}(S_A) \to R$ follows from the following equalities:

\[
\begin{align*}
\frac{\partial f}{\partial x_j}(b^r b^{-\varepsilon}) &= \frac{\partial f}{\partial x_j}(b^0) = 0, \\
\frac{\partial f}{\partial x_j}(a \triangleleft b) &= \frac{\partial f}{\partial x_j}(b^{-1} ab), \\
\frac{\partial f}{\partial x_j}(x \triangleleft a^0) &= \frac{\partial f}{\partial x_j}(x), \\
\frac{\partial f}{\partial x_j}(x \triangleleft (ab)) &= \frac{\partial f}{\partial x_j}((x \triangleleft a) \triangleleft b), \\
\frac{\partial f}{\partial x_j}((x \triangleleft y) \triangleleft z) &= \frac{\partial f}{\partial x_j}((x \triangleleft z) \triangleleft (y \triangleleft z)), \\
\frac{\partial f}{\partial x_j}((ab) \triangleleft x) &= \frac{\partial f}{\partial x_j}((a \triangleleft x) (b \triangleleft x)),
\end{align*}
\]

where $\varepsilon \in \{0, 1, -1\}$ and $a, b, c, x, y, z \in W_{MCQ}(S_A)$ such that $a, b, c$ are multiplicable. We can see that these equalities hold by direct calculation, where the fourth and sixth equalities follow from [20, Lemma 2.6]. This completes the proof. □

**Proposition 5.3.** For any $\mu \in \mathcal{K}$, $a \in G_\mu$ and $j \in \{1, \ldots, n\}$, we have

\[
\begin{align*}
\frac{\partial f}{\partial x_j}(e_\mu) &= 0, \\
\frac{\partial f}{\partial x_j}(a^{-1}) &= -f_1(a, a) \frac{\partial f}{\partial x_j}(a).
\end{align*}
\]

*Proof.* The first equality follows from

\[
\frac{\partial f}{\partial x_j}(e_\mu e_\mu) = \frac{\partial f}{\partial x_j}(e_\mu) + f_1(e_\mu, e_\mu) \frac{\partial f}{\partial x_j}(e_\mu) = \frac{\partial f}{\partial x_j}(e_\mu) + \frac{\partial f}{\partial x_j}(e_\mu).
\]

The second equality follows from

\[
\frac{\partial f}{\partial x_j}(e_\mu) = \frac{\partial f}{\partial x_j}(a^{-1} a) = \frac{\partial f}{\partial x_j}(a^{-1}) + f_1(a^{-1}, a) \frac{\partial f}{\partial x_j}(a).
\]

□

6. $f$-twisted Alexander matrices for handlebody-links

Let $R$ be a ring. We denote by $M(m, n; R)$ the set of $m \times n$ matrices over $R$ and by $GL(n; R)$ the set of $n \times n$ invertible matrices over $R$. We say that two matrices $A_1$ and $A_2$ over $R$ are equivalent, denoted by $A_1 \sim A_2$, if they are related by a finite sequence of the following transformations:

- $(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n) \leftrightarrow (a_1, \ldots, a_i + ar, \ldots, a_{i+1}, \ldots, a_n)$ ($r \in R$),
Let an MCQ Alexander pair by Proposition 3.10. For a relator \( r \) equivalent matrices are equivalent as flat matrices. Suppose that \( R \) be a commutative ring, and let \( A \in M(m, n; R) \). A \( k \)-minor of \( A \) is the determinant of a \( k \times k \) submatrix of \( A \). For any \( d \in \mathbb{Z}_{\geq 0} \), the \( d \)-th elementary ideal \( E_d(A) \) of \( A \) is the ideal of \( R \) generated by all \( (n-d) \)-minors of \( A \) if \( n-m \leq d < n \), and

\[
E_d(A) := \begin{cases} 
0 & \text{if } d < n - m, \\
R & \text{if } n - d.
\end{cases}
\]

Suppose that \( R \) is a GCD domain. Then the \( d \)-th Alexander invariant \( \Delta_d(A) \) of \( A \) is the greatest common divisor of all \( (n-d) \)-minors of \( A \) if \( n - m \leq d < n \), and

\[
\Delta_d(A) := \begin{cases} 
0 & \text{if } d < n - m, \\
1 & \text{if } n - d.
\end{cases}
\]

We remark that \( \Delta_d(A) \) coincides with the greatest common divisor of generators of \( E_d(A) \) and is determined up to unit multiple. If \( A \sim B \), then \( E_d(A) = E_d(B) \) and \( \Delta_d(A) = \Delta_d(B) \), where “\( \sim \)” means “is equal to, up to multiplication by a unit”. See [7] for more details.

**Remark 6.1.** Let \( R \) be a commutative ring. We can regard a matrix in \( M(m, n; M(k, k; R)) \) as a matrix in \( M(km, kn; R) \). We call such matrices flat matrices. We note that equivalent matrices are equivalent as flat matrices.

For an MCQ representation \( \rho : X \rightarrow Y \) and an MCQ Alexander pair \( f = (f_1, f_2) \) of maps \( f_1, f_2 : Y \times Y \rightarrow R \), we set \( f \circ (\rho \times \rho) := (f_1 \circ (\rho \times \rho), f_2 \circ (\rho \times \rho)) \), which is an MCQ Alexander pair by Proposition 3.10. For a relator \( r = (r_1, r_2) \), we define

\[
\frac{\partial f}{\partial x_j}(r) := \frac{\partial f}{\partial x_j}(r_1) - \frac{\partial f}{\partial x_j}(r_2).
\]

**Definition 6.2.** Let \( X = \langle \mathbf{x} | r \rangle = \langle x_1, \ldots, x_k ; x_{k+1}, \ldots, x_n | r_1, \ldots, r_m \rangle \) be a finitely presented MCQ and \( \rho : X \rightarrow Y \) an MCQ representation. Let \( f = (f_1, f_2) \)
be an MCQ Alexander pair of maps $f_1, f_2 : Y \times Y \to R$. The $f$-twisted Alexander matrix of $(X, \rho)$ (with respect to the presentation $(x \mid r)$) is

$$A(X, \rho; f_1, f_2) = \begin{pmatrix}
  \frac{\partial f_i(x, r)}{\partial x_1}(r_1) & \cdots & \frac{\partial f_i(x, r)}{\partial x_n}(r_1) \\
  \vdots & \ddots & \vdots \\
  \frac{\partial f_i(x, r)}{\partial x_1}(r_m) & \cdots & \frac{\partial f_i(x, r)}{\partial x_n}(r_m)
\end{pmatrix}.
$$

The following proposition shows that the equivalence class of an $f$-twisted Alexander matrix is invariant under the transformations (T1-1)–(T1-5), (T2) and (T3-1) with MCQ representations.

**Proposition 6.3.** Let $X = (x \mid r)$ and $X' = (x' \mid r')$ be finitely presented MCQs, and let $\rho : X \to Y$ and $\rho' : X' \to Y$ be MCQ representations. Let $(f_1, f_2)$ be an MCQ Alexander pair of maps $f_1, f_2 : Y \times Y \to R$. If $(X, \rho) \sim_T (X', \rho')$, then we have

$$A(X, \rho; f_1, f_2) \sim A(X', \rho'; f_1, f_2).$$

Especially, we have

$$E_d(A(X, \rho; f_1, f_2)) = E_d(A(X', \rho'; f_1, f_2))$$

if $R$ is a commutative ring, and we have

$$\Delta_d(A(X, \rho; f_1, f_2)) = \Delta_d(A(X', \rho'; f_1, f_2))$$

if $R$ is a GCD domain.

**Proof.** Put $f := (f_1, f_2)$ and $f^i := f_i \circ (\rho \times \rho)$ for each $i = 1, 2$, that is, $f \circ (\rho \times \rho) = (f^1, f^2)$. We show that the equivalence class of an $f$-twisted Alexander matrix $A(X, \rho, f_1, f_2)$ is invariant under the transformations (T1-1)–(T1-5), (T2) and (T3-1). We remark again that the same symbol $\rho$ is used for $\rho \circ f^1_{-1}$, $\rho \circ f^2_{2-1}$ and $\rho \circ f^1_{-1}$ on the transformations. We set $A := A((x \mid r), \rho; f_1, f_2)$ and $A' := A((x' \mid r'), \rho'; f_1, f_2)$. We denote by $a_i$ the $i$-th row vector of $A$, and we denote by $a_{i,j}$ (resp. $a'_{i,j}$) the $(i, j)$ entry of $A$ (resp. $A'$).

At first, we check (T1-1)–(T1-5). For the presentations

$$(x \mid r) = (x_1, \ldots, x_k; \ldots; x_1, \ldots, x_n \mid r_1, \ldots, r_m),$$

$$(x' \mid r') = (x_1, \ldots, x_k; \ldots; x_1, \ldots, x_n \mid r_1, \ldots, r_m, x = x) \ (x \in F_{MCQ}(x)),$$ we have

$$A \sim \begin{pmatrix}
  \frac{\partial f_i(x, r)}{\partial x_1}(r_1) & \cdots & \frac{\partial f_i(x, r)}{\partial x_n}(r_1) \\
  \vdots & \ddots & \vdots \\
  \frac{\partial f_i(x, r)}{\partial x_1}(r_m) & \cdots & \frac{\partial f_i(x, r)}{\partial x_n}(r_m)
\end{pmatrix} = A'.
$$

For the presentations

$$(x \mid r) = (x_1, \ldots, x_k; \ldots; x_1, \ldots, x_n \mid r_1, \ldots, r_m, a = b),$$

$$(x' \mid r') = (x_1, \ldots, x_k; \ldots; x_1, \ldots, x_n \mid r_1, \ldots, r_m, a = b, b = a),$$ we have

$$A = \begin{pmatrix}
a_1 \\
\vdots \\
a_m+1
\end{pmatrix} \sim \begin{pmatrix}
a_1 \\
\vdots \\
a_m+1
\end{pmatrix} \sim \begin{pmatrix}
a_1 \\
\vdots \\
a_m+1
\end{pmatrix} = A'.$
For the presentations
\[ \langle x | r \rangle = \langle x_1, \ldots, x_k; \ldots; x_l, \ldots, x_n | r_1, \ldots, r_m, a = b, b = c \rangle, \]
\[ \langle x' | r' \rangle = \langle x_1, \ldots, x_k; \ldots; x_l, \ldots, x_n | r_1, \ldots, r_m, a = b, b = c, a = c \rangle, \]
we have
\[ A = \begin{pmatrix} a_1 \\ \vdots \\ a_m+1 \\ a_{m+2} \end{pmatrix} \sim \begin{pmatrix} a_1 \\ \vdots \\ a_m+1 \\ a_{m+2} \end{pmatrix} \sim \begin{pmatrix} a_1 \\ \vdots \\ a_m+1 \\ a_{m+2} \end{pmatrix} = A'. \]

For the presentations
\[ \langle x | r \rangle = \langle x_1, \ldots, x_k; \ldots; x_l, \ldots, x_n | r_1, \ldots, r_m, a_1 = a_2, b_1 = b_2 \rangle, \]
\[ \langle x' | r' \rangle = \langle x_1, \ldots, x_k; \ldots; x_l, \ldots, x_n | r_1, \ldots, r_m, a_1 = a_2, b_1 = b_2, a_1 \prec b_1 = a_2 \prec b_2 \rangle, \]
we have
\[ A = \begin{pmatrix} a_1 \\ \vdots \\ a_m+1 \\ a_{m+2} \end{pmatrix} \sim \begin{pmatrix} a_1 \\ \vdots \\ a_m+1 \\ a_{m+2} \end{pmatrix} \sim \begin{pmatrix} a_1 \\ \vdots \\ a_m+1 \\ a_{m+2} \end{pmatrix} = A'. \]

For the presentations
\[ \langle x | r \rangle = \langle x_1, \ldots, x_k; \ldots; x_l, \ldots, x_n | r_1, \ldots, r_m, a_1 = a_2, b_1 = b_2 \rangle, \]
\[ \langle x' | r' \rangle = \langle x_1, \ldots, x_k; \ldots; x_l, \ldots, x_n | r_1, \ldots, r_m, a_1 = a_2, b_1 = b_2, a_1 b_1^{-1} = a_2 b_2^{-1} \rangle, \]
where \( a_i \) and \( b_i \) are multiplicable, we have
\[ A = \begin{pmatrix} a_1 \\ \vdots \\ a_m+1 \\ a_{m+2} \end{pmatrix} \sim \begin{pmatrix} a_1 \\ \vdots \\ a_m+1 \\ a_{m+2} \end{pmatrix} \sim \begin{pmatrix} a_1 \\ \vdots \\ a_m+1 \\ a_{m+2} \end{pmatrix} = A'. \]

Next, we check (T2). For the presentations
\[ \langle x | r \rangle = \langle x_1, \ldots, x_k; \ldots; x_l, \ldots, x_n | r_1, \ldots, r_m \rangle, \]
\[ \langle x' | r' \rangle = \langle x_1, \ldots, x_k; \ldots; x_l, \ldots, x_n; y | r_1, \ldots, r_m, y = w \rangle \quad (y \notin F_{MCQ}(x), w \in F_{MCQ}(x)), \]
we have
\[ A = \begin{pmatrix} \frac{\partial f_1(x, y)}{\partial x_1}(r_1) & \ldots & \frac{\partial f_1(x, y)}{\partial x_n}(r_1) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m(x, y)}{\partial x_1}(r_m) & \ldots & \frac{\partial f_m(x, y)}{\partial x_n}(r_m) \end{pmatrix} \sim \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} A & 0 \\ -\frac{\partial w}{\partial x} & 1 \end{pmatrix} = A', \]
where \(-\frac{\partial w}{\partial x}\) is the row vector \( \left( -\frac{\partial f_1(x, y)}{\partial x}(w), \ldots, -\frac{\partial f_m(x, y)}{\partial x}(w) \right) \).
Finally, we check (T3-1). For the presentations
\[
\langle x | r \rangle = \langle x_1, \ldots, x_k; \ldots; x_1, \ldots, x_{l+1}, \ldots, x_n | r_1, \ldots, r_m, a^0 = b^0 \rangle,
\]
\[
\langle x' | r' \rangle = \langle x_1, \ldots, x_k; \ldots; x'_1, \ldots, x_n | r_1, \ldots, r_m, a^0 = b^0 \rangle,
\]
where \( a \in \{ x_1, \ldots, x_l \} \) and \( b \in \{ x_{l+1}, \ldots, x_n \} \), we have \( A = A' \) easily.

This completes the proof. \( \square \)

Let \( H \) be a handlebody-link represented by a Y-oriented diagram \( D \). Let \( \rho : \text{MCQ}(D) \to X \) be an MCQ representation, which can be regarded as an \( X \)-coloring of \( D \). Let \( f = (f_1, f_2) \) be an MCQ Alexander pair of maps \( f_1, f_2 : X \times X \to R \). Then we define the \( f \)-twisted Alexander matrix of \( (H, \rho) \) (with respect to \( D \)) by
\[
A(H, \rho; f_1, f_2) = A(\text{MCQ}(D), \rho; f_1, f_2).
\]

We also define
\[
E_d(H, \rho; f_1, f_2) := E_d(A(\text{MCQ}(D), \rho; f_1, f_2)),
\]
\[
\Delta_d(H, \rho; f_1, f_2) := \Delta_d(A(\text{MCQ}(D), \rho; f_1, f_2))
\]
if \( R \) is a commutative ring or a GCD domain, respectively.

By Lemma 4.2 and Proposition 6.3, we have the following theorem.

**Theorem 6.4.** Let \( H \) and \( H' \) be handlebody-links represented by Y-oriented diagrams \( D \) and \( D' \), respectively. Let \( \rho : \text{MCQ}(D) \to X \) and \( \rho' : \text{MCQ}(D') \to X \) be MCQ representations. Let \( (f_1, f_2) \) be an MCQ Alexander pair of maps \( f_1, f_2 : X \times X \to R \). If \( (H, \rho) \cong (H', \rho') \), then we have
\[
A(H, \rho; f_1, f_2) \sim A(H', \rho'; f_1, f_2).
\]
Especially, we have
\[
E_d(H, \rho; f_1, f_2) = E_d(H', \rho'; f_1, f_2)
\]
if \( R \) is a commutative ring, and we have
\[
\Delta_d(H, \rho; f_1, f_2) \cong \Delta_d(H', \rho'; f_1, f_2)
\]
if \( R \) is a GCD domain.

We calculate our invariants of the genus \( g \) trivial handlebody-knot for any MCQ Alexander pair \( f \).

**Proposition 6.5.** Let \( O_g \) be the trivial handlebody-knot of genus \( g \). Let \( D_g \) be the Y-oriented diagram of \( O_g \) illustrated in Figure 5. For any MCQ representation \( \rho : \text{MCQ}(D_g) \to X \) and MCQ Alexander pair \( (f_1, f_2) \) of maps \( f_1, f_2 : X \times X \to R \), we have
\[
A(O_g, \rho; f_1, f_2) \sim (0 \cdots 0) \in M(1, g; R).
\]
Especially, we have
\[
E_d(O_g, \rho; f_1, f_2) = \begin{cases} 0 & \text{if } d < g, \\ R & \text{if } g \leq d \end{cases}
\]
if \( R \) is a commutative ring, and we have
\[
\Delta_d(O_g, \rho; f_1, f_2) = \begin{cases} 0 & \text{if } d < g, \\ 1 & \text{if } g \leq d \end{cases}
\]
if \( R \) is a GCD domain.
Proof. Assume that \( g \geq 2 \). The Wirtinger presentation \( \text{MCQ}(D_g) \) is given by

\[
\begin{aligned}
x_1, \ldots, x_{3g-3} & \mid x_1x_2 = x_1, \ x_{3g-3}x_{3g-4} = x_{3g-3}, \\
x_{3i}x_{3i+1} = x_{3i-1}, \ x_{3i}x_{3i+1} = x_{3i+2} \ (1 \leq i \leq g-2) \end{aligned}
\]

We put \( a_i := f_1 \circ (\rho \times \rho)(x_i, x_i^{-1}) \in R^\times \). We then have

\[
\begin{array}{cccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc
\end{array}
\]

\[
= \begin{pmatrix}
0 & a_1 & 1 & a_3 & \cdots & 0 \\
0 & -1 & 1 & a_3 & -1 & \cdots \\
1 & a_3 & -1 & 1 & a_6 & -1 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & -1 & 1 & a_{3g-6} & -1 \\
0 & \cdots & 1 & a_{3g-6} & \cdots & a_{3g-3}
\end{pmatrix}
\]

\sim (0 \cdots 0) \in M(1,g;R).

It is easy to see that \( A(O_1, \rho; f_1, f_2) \sim (0) \in M(1,1;R) \).

Remark 6.6. In the same way as proof of Proposition 6.5, we have the following statement. Let \( O_{g_1, \ldots, g_m} \) be the \( m \)-component trivial handlebody-link the genera of whose components are \( g_1, \ldots, g_m \), respectively. Let \( D_{g_1, \ldots, g_m} \) be a Y-oriented diagram of \( O_{g_1, \ldots, g_m} \). Put \( g := g_1 + \cdots + g_m \). For any MCQ representation \( \rho : \text{MCQ}(D_{g_1, \ldots, g_m}) \to X \) and MCQ Alexander pair \((f_1, f_2)\) of maps \( f_1, f_2 : X \times X \to M(n,n;R) \), we have

\[
A(O_{g_1, \ldots, g_m}, \rho; f_1, f_2) \sim (0 \cdots 0) \in M(1,ng;R),
\]

where we regard \( A(O_{g_1, \ldots, g_m}, \rho; f_1, f_2) \) as a flat matrix. Especially, we have

\[
E_d(O_{g_1, \ldots, g_m}, \rho; f_1, f_2) = \begin{cases} 
0 & \text{if } d < ng, \\
R & \text{if } ng \leq d
\end{cases}
\]

if \( R \) is a commutative ring, and we have

\[
\Delta_d(O_{g_1, \ldots, g_m}, \rho; f_1, f_2) = \begin{cases} 
0 & \text{if } d < ng, \\
1 & \text{if } ng \leq d
\end{cases}
\]

if \( R \) is a GCD domain.
7. (Twisted) Alexander matrices for handlebody-links

In this section, we recall the notion of the (twisted) Alexander matrix for handlebody-links [15] and see that it can be realized as an \( f \)-twisted Alexander matrix for some MCQ Alexander pair \( f \).

Let \( H \) be a handlebody-link represented by a Y-oriented diagram \( D \). We then have the Wirtinger presentations

\[
\begin{align*}
\text{MCQ}(D) & = \langle x_1, \ldots, x_k; \ldots; x_l, \ldots, x_n \mid r_1, \ldots, r_m \rangle, \\
G(D) & = \langle x_1, \ldots, x_k, \ldots, x_l, \ldots, x_n \mid r_{1}^{\text{Grp}}, \ldots, r_{m}^{\text{Grp}} \rangle,
\end{align*}
\]

where

\[
\begin{align*}
r_i & = \begin{cases} (u_i < v_i, w_i) & \text{if } 1 \leq i \leq m', \\
(u_i v_i, w_i) & \text{if } m' + 1 \leq i \leq m,
\end{cases} \\
r_i^{\text{Grp}} & = \begin{cases} v_i^{-1} u_i v_i w_i^{-1} & \text{if } 1 \leq i \leq m', \\
u_i v_i w_i^{-1} & \text{if } m' + 1 \leq i \leq m.
\end{cases}
\end{align*}
\]

See Section 4. Let \( R \) be a commutative ring. Set \( G := GL(k; R) \). For a group representation \( \rho : G(D) \rightarrow G \), we use the same symbol \( \rho \) for the induced MCQ representation of \( \text{MCQ}(D) \) to \( G \) sending \( x_i \) into \( \rho(x_i) \), where we regard \( G \) as an MCQ with the conjugation operation. Put \( S := \{ x_1, \ldots, x_n \} \), and let \( F_{\text{Grp}}(S) \) be the free group on \( S \) and \( G_0 \) the abelian group

\[
\left\langle t_1, \ldots, t_r \mid t_1^{k_1}, \ldots, t_r^{k_r}, [t_i, t_j] \mid 1 \leq i < j \leq r \right\rangle,
\]

where \( k_1, \ldots, k_r \in \mathbb{Z}_{\geq 0} \). The group ring \( R[G_0] \) can be identified with the quotient ring of the Laurent polynomial ring \( R[t_1^{\pm 1}, \ldots, t_r^{\pm 1}]/(t_1^{\pm 1} - 1, \ldots, t_r^{\pm 1} - 1) \). The Fox derivative [6] with respect to \( x_j \) is the \( R \)-homomorphism

\[
\frac{\partial_{\text{Grp}}}{\partial x_j} : R[F_{\text{Grp}}(S)] \rightarrow R[F_{\text{Grp}}(S)]
\]

satisfying

\[
\frac{\partial_{\text{Grp}}}{\partial x_j}(pq) = \frac{\partial_{\text{Grp}}}{\partial x_j}(p) + p \frac{\partial_{\text{Grp}}}{\partial x_j}(q) \quad \text{and} \quad \frac{\partial_{\text{Grp}}}{\partial x_j}(x_i) = \delta_{ij}.
\]

Let \( \text{pr} : F_{\text{Grp}}(S) \rightarrow G(D) \) be the canonical projection and \( \alpha : G(D) \rightarrow G_0 \) an surjective homomorphism. We denote the linear extensions of \( \text{pr} \) and \( \rho \) by the same symbols \( \text{pr} : R[F_{\text{Grp}}(S)] \rightarrow R[G(D)], \alpha : R[G(D)] \rightarrow R[G_0] \) and \( \rho : R[G(D)] \rightarrow M(k, k; R[G_0]) \), respectively. Then the \( R \)-homomorphism

\[
\rho \otimes \alpha : R[G(D)] \rightarrow M(k, k; R[G_0])
\]

is defined by

\[
(\rho \otimes \alpha) \left( \sum r_i g_i \right) = \sum r_i \rho(g_i) \alpha(g_i) \quad (r_i \in R, g_i \in G(D)).
\]

The Alexander matrix of \( G(D) \) associated with \( \alpha \) is the \( m \times n \) matrix

\[
\left( \alpha \circ \text{pr} \right) \left( \frac{\partial_{\text{Grp}}}{\partial x_j} (r_i^{\text{Grp}}) \right)
\]
over \( R[G_0] \). The twisted Alexander matrix of \( G(D) \) associated with \( \alpha \) and \( \rho \) is the matrix
\[
\left( (\rho \otimes \alpha) \circ \text{pr} \left( \frac{\partial_{\text{Grp}}}{\partial x_j} (r^\iota) \right) \right),
\]
which we regard as a \( km \times kn \) matrix over \( R[G_0] \). These matrices produce invariants for handlebody-links by evaluating them with ideals and greatest common divisors as we see in [15].

For the induced MCQ representations \( \alpha : \text{MCQ}(D) \to G_0 \) and \( \rho : \text{MCQ}(D) \to G \), we define the MCQ representation \( \rho \cdot \alpha : \text{MCQ}(D) \to GL(k; R[G_0]) \) by \( (\rho \cdot \alpha)(x) = \rho(x) \alpha(x) \). Under the following proposition, the (twisted) Alexander matrices for handlebody-links can be obtained in our framework.

**Proposition 7.1.** (1) Let \( f = (f_1, f_2) \) be the MCQ Alexander pair in Example 3.2, that is, \( f_1, f_2 : G_0 \times G_0 \to R[G_0] \) defined by \( f_1(a, b) = b^{-1} \) and \( f_2(a, b) = b^{-1}a - b^{-1} \). Let \( \alpha : G(D) \to G_0 \) be a surjective homomorphism. Then the \( f \)-twisted Alexander matrix \( A(H; \alpha; f_1, f_2) \) coincides with the Alexander matrix of \( G(D) \) associated with \( \alpha \).

(2) Set \( G := GL(k; R) \). Let \( f = (f_1, f_2) \) be the MCQ Alexander pair in Example 3.3, that is, \( f_1, f_2 : GL(k, R[G_0]) \times GL(k, R[G_0]) \to M(k; k; R[G_0]) \) defined by \( f_1(a, b) = b^{-1} \) and \( f_2(a, b) = b^{-1}a - b^{-1} \). Let \( \alpha : G(D) \to G_0 \) be a surjective homomorphism and \( \rho : G(D) \to G \) a group representation. Then the \( f \)-twisted Alexander matrix \( A(H, \rho \cdot \alpha; f_1, f_2) \) coincides with the twisted Alexander matrix of \( G(D) \) associated with \( \alpha \) and \( \rho \).

**Proof.** We prove (2). Put \( f_i^\rho := f_i \circ (\rho \cdot \alpha \times \rho \cdot \alpha) \) for each \( i = 1, 2 \), that is, \( f \circ (\rho \cdot \alpha \times \rho \cdot \alpha) = (f_1^\rho, f_2^\rho) \). We remark again that we often omit “pr” to represent \( \text{pr}(x) \) as \( x \). We note that \( \frac{\partial_{\text{Grp}}}{\partial x_j}(u_i), \frac{\partial_{\text{Grp}}}{\partial x_j}(v_i) \) and \( \frac{\partial_{\text{Grp}}}{\partial x_j}(w_i) \) are equal to 0 or 1 for each \( i \) and \( j \). For \( 1 \leq i \leq m' \), we have

\[
\left( (\rho \otimes \alpha) \circ \text{pr} \left( \frac{\partial_{\text{Grp}}}{\partial x_j} (v_i^{-1}u_iw_i^{-1}) \right) \right)
= \left( (\rho \otimes \alpha) \circ \text{pr} \right) \left( \left( u_i^{-1}u_i^{-1}v_i^{-1}u_iw_i^{-1} \right) + v_i^{-1}u_i^{-1}u_iw_i^{-1}u_iw_i^{-1} \right)
= \left( (\rho \otimes \alpha) \circ \text{pr} \right) \left( \left( u_i^{-1}v_i^{-1}u_iw_i^{-1}u_iw_i^{-1} \right) + v_i^{-1}u_i^{-1}u_iw_i^{-1}u_iw_i^{-1} \right)
= \left( f_1^\rho(a, v_i) \frac{\partial_{\text{Grp}}}{\partial x_j} (u_i) \right) + f_2^\rho(a, v_i) \frac{\partial_{\text{Grp}}}{\partial x_j} (v_i) - \frac{\partial_{\text{Grp}}}{\partial x_j} (u_i)
= \frac{\partial_{\text{Grp}}}{\partial x_j} (u_i) \cdot \frac{\partial_{\text{Grp}}}{\partial x_j} (u_i) \cdot \frac{\partial_{\text{Grp}}}{\partial x_j} (v_i) - \frac{\partial_{\text{Grp}}}{\partial x_j} (u_i)
= \frac{\partial_{\text{Grp}}}{\partial x_j} (u_i) \cdot \frac{\partial_{\text{Grp}}}{\partial x_j} (u_i) \cdot \frac{\partial_{\text{Grp}}}{\partial x_j} (v_i) - \frac{\partial_{\text{Grp}}}{\partial x_j} (u_i)
= \frac{\partial_{\text{Grp}}}{\partial x_j} (u_i) \cdot \frac{\partial_{\text{Grp}}}{\partial x_j} (u_i) \cdot \frac{\partial_{\text{Grp}}}{\partial x_j} (v_i) - \frac{\partial_{\text{Grp}}}{\partial x_j} (u_i)
= \frac{\partial_{\text{Grp}}}{\partial x_j} (u_i) \cdot \frac{\partial_{\text{Grp}}}{\partial x_j} (u_i) \cdot \frac{\partial_{\text{Grp}}}{\partial x_j} (v_i) - \frac{\partial_{\text{Grp}}}{\partial x_j} (u_i)
For $m' + 1 \leq i \leq m$, we have

\[
((\rho \otimes \alpha) \circ \text{pr}) \left( \frac{\partial_{\text{Grp}}}{\partial x_j}(u_i v_i w_i^{-1}) \right) \\
= ((\rho \otimes \alpha) \circ \text{pr}) \left( \frac{\partial_{\text{Grp}}}{\partial x_j}(u_i) + u_i \frac{\partial_{\text{Grp}}}{\partial x_j}(v_i) - u_i v_i w_i^{-1} \frac{\partial_{\text{Grp}}}{\partial x_j}(w_i) \right) \\
= \frac{\partial_{\text{Grp}}}{\partial x_j}(u_i) + ((\rho \otimes \alpha) \circ \text{pr})(u_i) \frac{\partial_{\text{Grp}}}{\partial x_j}(v_i) - \frac{\partial_{\text{Grp}}}{\partial x_j}(w_i) \\
= \frac{\partial f_{\circ (\rho \circ \rho \circ \rho \circ \rho)}}{\partial x_j}(u_i) + f_{i}^{\rho \circ \rho}(u_i, u_i^{-1}) \frac{\partial f_{\circ (\rho \circ \rho \circ \rho \circ \rho)}}{\partial x_j}(v_i) - \frac{\partial f_{\circ (\rho \circ \rho \circ \rho \circ \rho)}}{\partial x_j}(w_i) \\
= \frac{\partial f_{\circ (\rho \circ \rho \circ \rho \circ \rho)}}{\partial x_j}(u_i v_i = w_i).
\]

Therefore the $f$-twisted Alexander matrix $A(H, \rho \cdot \alpha; f_1, f_2)$ coincides with the twisted Alexander matrix of $G(D)$ associated with $\alpha$ and $\rho$.

In the same way, we can prove (1). \hfill \Box

8. $f$-TWISTED AXELROD MATERICES AND HANDLEBODY-KNOT COMPLEMENTS

In this section, calculating our invariants, we distinguish two handlebody-knots whose complements have isomorphic fundamental groups, which can not be distinguished by invariants derived from the (twisted) Alexander matrices. That is, $f$-twisted Alexander matrices produce strictly stronger invariants than (twisted) Alexander matrices for handlebody-knots.

**Example 8.1.** Let $H_1$ and $H_2$ be the handlebody-knots represented by the Y-oriented diagrams $D_1$ and $D_2$ depicted in Figure 6, respectively. We note that $H_1$ and $H_2$ have complements whose fundamental groups are isomorphic. The Wirtinger presentations $\text{MCQ}(D_1)$ and $\text{MCQ}(D_2)$ are given by

\[
\begin{align*}
& x_1, x_{10}, x_{11}, x_7, x_8, x_{12}; \\
& x_2; x_3; x_4; x_5; x_6; x_9 \quad | \quad x_1 \sqcup x_6 = x_2, x_2 \sqcup x_7 = x_3, x_3 \sqcup x_5 = x_4, x_4 \sqcup x_3 = x_5, \\
& x_5 \sqcup x_4 = x_6, x_6 \sqcup x_2 = x_7, x_8 \sqcup x_{11} = x_9, x_9 \sqcup x_{12} = x_{10}, \\
& x_{11} \sqcup x_9 = x_{12}, x_{11}x_{11} = x_{10}, x_{12}x_7 = x_8
\end{align*}
\]

and

\[
\begin{align*}
& x_1, x_{10}, x_{11}, x_7, x_8, x_{12}; \\
& x_2; x_3; x_4; x_5; x_6; x_9 \quad | \quad x_1 \sqcup x_6 = x_2, x_2 \sqcup x_7 = x_3, x_3 \sqcup x_5 = x_4, x_4 \sqcup x_3 = x_5, \\
& x_5 \sqcup x_4 = x_6, x_6 \sqcup x_2 = x_7, x_8 \sqcup x_{11} = x_9, x_9 \sqcup x_{12} = x_{10}, \\
& x_{11} \sqcup x_9 = x_{12}, x_{11}x_{11} = x_{10}, x_{12}x_7 = x_8
\end{align*}
\]

respectively. Let $X$ and $(\tilde{f}_1, 0)$ be the MCQ and the MCQ Alexander pair of maps $\tilde{f}_1 : X \times X \to Z_4[\{\pm 1\}]/(t^4 - 1)$ in Example 3.8, respectively. Putting $a_{\rho} := \tilde{f}_1 \circ (\rho \times \rho)(u_\xi, v_\xi)$ for each MCQ representation $\rho : \text{MCQ}(D_1) \to X$ and
each crossing \( c_i \) of \( D_1 \), we have

\[
A(H_1, \rho; \tilde{f}_1, 0) = \begin{pmatrix}
\alpha^0_1 & -1 & -1 \\
\alpha^0_2 & \alpha^0_3 & -1 \\
\alpha^0_4 & \alpha^0_5 & -1 \\
\alpha^0_6 & -1 \\
1 & 1 & -1 & 1
\end{pmatrix}
\]

\[
\sim (a_1^0 a_2^0 a_3^0 a_4^0 a_5^0 a_6^0 \alpha^0_1 a_7^0 a_8^0 a_9^0 - 1) \in M(1, 2; \mathbb{Z}_4[t^\pm 1]/(t^4 - 1)).
\]

On the other hand, putting \( b_\rho^0 := e \circ (\rho \times \rho)(u_{c_i}, v_{c_i}) \) for each MCQ representation \( \rho : MCQ(D_2) \rightarrow X \) and each crossing \( c_i \) of \( D_2 \), we have

\[
A(H_2, \rho; \tilde{f}_1, 0) = \begin{pmatrix}
b_1^0 & -1 & -1 \\
-1 & b_2^0 & b_3^0 \\
-1 & b_4^0 & b_5^0 & -1 \\
1 & 1 & -1 & 1
\end{pmatrix}
\]

\[
\sim (b_1^0 b_2^0 b_3^0 - b_4^0 b_5^0 b_6^0 b_7^0 b_8^0 b_9^0 - 1) \in M(1, 2; \mathbb{Z}_4[t^\pm 1]/(t^4 - 1)).
\]

Evaluating these matrices for each MCQ representation \( \rho \), we obtain the multisets

\[
\{ \Delta_1 \left( H_1, \rho; \tilde{f}_1, 0 \right) \mid \rho \in \text{Hom}(MCQ(D_1), X) \} = \bigg\{ 0 \text{ (96 times)}, t + 3 \text{ (384 times)},
\]

\[
t^2 + 3 \text{ (768 times)} \bigg\}
\]

and

\[
\{ \Delta_1 \left( H_2, \rho; \tilde{f}_1, 0 \right) \mid \rho \in \text{Hom}(MCQ(D_2), X) \} = \{ 0 \text{ (864 times)}, t + 3 \text{ (384 times)} \}.
\]

Therefore \( H_1 \) and \( H_2 \) are not equivalent by Theorem 6.4.

**Remark 8.2.** We emphasize again that, in Example 8.1, \( H_1 \) and \( H_2 \) have complements whose fundamental groups are isomorphic. This implies that \( f \)-twisted Alexander matrices yield strictly stronger invariants than (twisted) Alexander matrices for handlebody-knots. In particular, it also indicates that the fundamental MCQ distinguishes handlebody-knots whose complements have isomorphic fundamental groups. Then it is natural to consider the following problem.

**Problem 8.3.** Does the fundamental MCQ distinguish handlebody-knots whose complements are homeomorphic? In particular, do our invariants distinguish them?
Figure 6. Y-oriented diagrams of two handlebody-knots whose complements have isomorphic fundamental groups.

We can easily see that the fundamental MCQ distinguishes handlebody-“links” (not handlebody-“knots”) whose complements are homeomorphic. Let $H_1$ and $H_2$ be the 2-component handlebody-links represented by the Y-oriented diagrams $D_1$ and $D_2$ depicted in Figure 7, respectively. We note that $H_1$ and $H_2$ have homeomorphic complements. Set

$$M := (a_{ij}) = \begin{pmatrix} 1 & 3 & 2 & 5 & 4 & 1 \\ 3 & 2 & 1 & 6 & 2 & 4 \\ 2 & 1 & 3 & 3 & 6 & 5 \\ 5 & 6 & 4 & 4 & 1 & 2 \\ 4 & 5 & 6 & 1 & 5 & 3 \\ 6 & 4 & 5 & 2 & 3 & 6 \end{pmatrix}$$

and $Q_M := \{1, 2, 3, 4, 5, 6\}$. We define a binary operation $\prec$ on $Q_M$ by $i \prec j = a_{ij}$. Then $(Q_M, \prec)$ is a quandle, called the quandle $QS_{61}$. Since type $Q_M = 2$, we have the associated MCQ $X := Q_M \times \mathbb{Z}_2$ of the $\mathbb{Z}_2$-family of quandles $(Q_M, \{c^i\}_{i \in \mathbb{Z}_2})$. Then we obtain

$$\#\text{Hom}(\text{MCQ}(D_1), X) = 192 \quad \text{and} \quad \#\text{Hom}(\text{MCQ}(D_2), X) = 168,$$

where $\#S$ denotes the cardinality of the set $S$. Hence we have $\text{MCQ}(D_1) \not\cong \text{MCQ}(D_2)$, which implies that $H_1$ and $H_2$ are not equivalent.

ACKNOWLEDGMENTS

This work was supported by JSPS KAKENHI Grant Numbers JP21K03217, JP20K22312, JP21K13796.
Figure 7. Y-oriented diagrams of two handlebody-links whose complements are homeomorphic.

REFERENCES


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